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E. L. Cooley

A TREATISE

ON

HYDRAULICS.

BY

MANSFIELD MERRIMAN,
PROFESSOR OF CIVIL ENGINEERING IN LEHIGH UNIVERSITY.

FOURTH AND REVISED EDITION.

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NEWTON.

HYDRAULICS.

CHAPTER I.

INTRODUCTION.

Article 1. Units of Measure.

The unit of linear measure universally adopted in English and American hydraulic literature is the foot, which is defined as one-third of the standard yard. For some minor purposes, such as the designation of the diameters of orifices and pipes, the inch is employed, but inches should always be reduced to feet for use in hydraulic formulas. The unit of superficial measure is usually the square foot, except for the expression of the intensity of pressures, when the square inch is more commonly employed.

The units of volume employed in measuring water are the cubic foot and the gallon. In Great Britain the Imperial gallon is used, and in this country the old English gallon, the former being 20 per cent larger than the latter. The following are the relations between the cubic foot and the two gallons:

1 cubic foot = 6.232 Imp. gallons = 7.481 U. S. gallons;
1 Imp. gallon = 0.1605 cubic feet = 1.200 U. S. gallons;
1 U. S. gallon = 0.1337 cubic feet = 0.8331 Imp. gallons.

In this book the word gallon will always mean the United States gallon of 231 cubic inches, unless otherwise stated.

The unit of weight is the avoirdupois pound, which is also the unit for measuring pressures. The intensity of pressure will be measured in pounds per square foot or in pounds per square inch, as may be most convenient, and sometimes in atmospheres (Art. 4). Gauges for recording the pressure of water are usually graduated so as to read pounds per square inch.

The unit of time used in all hydraulic formulas is the second, although in numerical problems the time is often stated in minutes, hours, or days. Velocity is defined as the space passed over by a body in one second under the condition of uniform motion, so that velocities are to be always expressed in feet per second, or are to be reduced to these units if stated in miles per hour or otherwise.

The unit of work, or energy, is the foot-pound; that is, one pound lifted through a vertical distance of one foot. Energy is potential work, or the work which can be done; for example, a moving stream of water has the ability to do a certain amount of work by virtue of its weight and velocity, and this is called energy, while the word work is more generally used for that actually done by a motor which is moved by the water. Power is work, or energy, done or existing in a specified time, and the unit for its measure is the horse-power, which is 550 foot-pounds per second, or 33 000 foot-pounds per minute.

In French and German literature the metric system is employed; the meter and centimeter being the units of length, and their squares the units of superficial measure. The units of capacity are the cubic meter and the liter, that of weight the kilogram, and that of time the second. The unit of work is the kilogram-meter, and one horse-power is 75 kilogram-meters, which is about 1.5 per cent less than that as defined above. Students should be prepared to rapidly transform metric into

American measures, for which purpose a table of equivalents giving logarithms will be found most convenient.*

The motion of water in river channels, and its flow through orifices and pipes, is produced by the force of gravity. This force is proportional to the acceleration of the velocity of a body falling freely in a vacuum; that is, to the increase in velocity in one second. The acceleration is measured in feet per second per second, so that its value represents the number of feet per second which have been gained in one second by a falling body.

Problem 1. How many pounds per square inch are equivalent to a pressure of 70 kilograms per square centimeter?

ARTICLE 2. PHYSICAL PROPERTIES OF WATER.

At ordinary temperatures pure water is a colorless liquid which possesses perfect fluidity; that is, its particles have the capacity of moving over each other, so that the slightest disturbance of equilibrium causes a flow. It is a consequence of this property that the surface

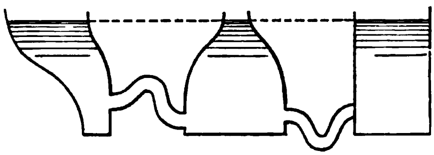

FIG. 1.

of still water is always level; also, if several vessels or tubes be connected, as in Fig. 1, and water be poured into one of them, it rises in the others until, when equilibrium ensues, the free surfaces are in the same level plane.

The free surface of water is in a different molecular condition from the other portions, its particles being drawn together by stronger attractive forces, so as to form what may be called the "skin of the water," upon which insects walk. The skin is not immediately pierced by a sharp point which moves slowly

* See LANDRETH's Metrical Tables for Engineers (Philadelphia, 1883).

upward toward it, but a slight elevation occurs, and this property enables precise determinations of the level of still water to be made by means of the hook gauge (Art. 50).

At about 32 degrees Fahrenheit a great alteration in the molecular constitution of water occurs, and ice is formed. If a quantity of water be kept in a perfectly quiet condition, it is found that its temperature can be reduced to 20°, or even to 15°, Fahrenheit, before congelation takes place, but at the moment when this occurs the temperature rises to 32°. The freezing-point is hence not constant, but the melting-point of ice is always at the same temperature of 32° Fahrenheit or 0° Centigrade.

Ice being lighter than water, forms as a rule upon its surface; but when water is in rapid motion a variety called anchor ice may occur. In this case the ice is formed at the surface in the shape of small needles, which are quickly carried to the lower strata by the agitation due to the motion; there the needles adhere to the bed of the stream, sometimes accumulating to an extent sufficient to raise the water level several feet.* Anchor ice frequently causes obstructions in conduits and orifices which lead water to motors.

Water is a solvent of high efficiency, and is therefore never found pure in nature. Descending in the form of rain it absorbs dust and gaseous impurities from the atmosphere; flowing over the surface of the earth it absorbs organic and mineral substances. These affect its weight only slightly as long as it remains fresh, but when it has reached the sea and become salt its weight is increased more than two per cent. The flow of water through orifices and pipes is only in a very slight degree affected by the impurities held in solution.

---

* FRANCIS in Transactions American Society Civil Engineers, 1881, vol. x. p. 192.

The capacity of water for heat, the latent heat evolved when it freezes, and that absorbed when it is transformed into steam, need not be considered for the purposes of hydraulic investigations. Other physical properties, such as its variation in volume with the temperature, its compressibility, and its capacity for transmitting pressures, are discussed in detail in the following pages. The laws which govern its pressure, flow, and energy under various circumstances belong to the science of Hydraulics, and form the subject-matter of this volume.

Prob. 2. What horse-power is required to lift 16000 pounds of water per minute through a vertical height of 21 feet? Ans. 10.2.

ARTICLE 3. THE WEIGHT OF WATER.

The weight of water per unit of volume depends upon the temperature and upon its degree of purity. The following approximate values are, however, those generally employed except when great precision is required:

1 cubic foot weighs 62.5 pounds;
1 U. S. gallon weighs 8.355 pounds.

These values will be used in this book, unless otherwise stated, in the solution of the examples and problems.

The weight per unit of volume of pure distilled water is the greatest at the temperature of its maximum density, 39°.3 Fahrenheit, and least at the boiling-point. For ordinary computations the variation in weight due to temperature is not considered, but in tests of the efficiency of hydraulic motors and of pumps it should be regarded. The following table is hence given, which contains the weights of one cubic foot of pure water at different temperatures as deduced by SMITH from the experiments of ROSSETTI.*

* HAMILTON SMITH, Jr., Hydraulics: The Flow of Water through Orifices, over Weirs, and through open Conduits and Pipes (London and New York, 1886), p. 14.

TABLE I. WEIGHT OF DISTILLED WATER.

Temperature (Fahrenheit).	Pounds per Cubic Foot.	Temperature (Fahrenheit).	Pounds per Cubic Foot.	Temperature (Fahrenheit).	Pounds per Cubic Foot.
32°	62.42	95	62.06	160	61.01
35	62.42	100	62.00	165	60.90
39.3	62.424	105	61.93	170	60.80
45	62.42	110	61.86	175	60.69
50	62.41	115	61.79	180	60.59
55	62.39	120	61.72	185	60.48
60	62.37	125	61.64	190	60.36
65	62.34	130	61.55	195	60.25
70	62.30	135	61.47	200	60.14
75	62.26	140	61.39	205	60.02
80	62.22	145	61.30	210	59.89
85	62.17	150	61.20	212	59.84
90	62.12	155	61.11		

Waters of rivers, springs, and lakes hold in suspension and solution inorganic matters which cause the weight per unit of volume to be slightly greater than for pure water. River waters are usually between 62.3 and 62.5 pounds per cubic foot, depending upon the amount of impurities and on the temperature, while the water of some mineral springs has been found to be as high as 62.7. It appears that, in the absence of specific information regarding a particular water, the weight 62.5 pounds per cubic foot is a fair approximate value to use. It also has the advantage of being a convenient number in computations, for 62.5 pounds is 1000 ounces, or $\frac{1000}{16}$ is the equivalent of 62.5.

In the metric system the weight of a cubic meter of pure water at a temperature near that of maximum density is taken as 1000 kilograms, which is the average unit-weight used in hydraulic computations. This corresponds to 62.426 pounds per cubic foot.

Brackish and salt waters are always much heavier than fresh water. For the Gulf of Mexico the weight per cubic foot is about 63.9, for the oceans about 64.1, while for the Dead Sea there is stated the value 73 pounds per cubic foot. The weight of ice per cubic foot varies from 57.2 to 57.5 pounds.

Prob. 3. How many pounds of water in a cylindrical box 2 feet in diameter and 2 feet deep? How many gallons? How many kilograms? How many liters?

Prob. 4. In a certain problem regarding the horse-power required to lift water, the computations were made with the mean value 62.5 pounds per cubic foot. Supposing that the actual weight per cubic foot was 62.35 pounds, show that the error thus introduced was less than one-fourth of one per cent.

ARTICLE 4. ATMOSPHERIC PRESSURE.

The pressure of the atmosphere is measured by the readings of the barometer. This instrument is a tube entirely exhausted of air, which is inserted into a vessel containing a liquid. The pressure of the air on the surface of the liquid causes it to rise in the tube until it attains a height which exactly balances the pressure of the air. Or in other words, the weight of the barometric column is equal to the weight of a column of air of the same cross-section as that of the tube, both columns being measured upward from the surface of the liquid in the vessel. The liquid generally employed is mercury, and, owing to its great density, the height of the column required to balance the atmospheric pressure is only about 30 inches, whereas a water barometer would require a height of over 30 feet.

The atmosphere exerts its pressure with varying intensity, as indicated by the readings of the mercury barometer. At and near the sea level the average reading is 30 inches, and as mercury weighs 0.49 pounds per cubic inch at common temperatures, the average atmospheric pressure is taken to be 30×0.49 or 14.7 pounds per square inch.

The pressure of one atmosphere is therefore defined to be a pressure of 14.7 pounds per square inch. Then a pressure of two atmospheres is 29.4 pounds per square inch. And con-

versely, a pressure of one pound per square inch may be expressed as a pressure of 0.068 atmospheres.

The rise of water in a vacuum is due merely to the pressure of the atmosphere, like that of the mercury in the common barometer. In a perfect vacuum, water will rise to a height of about 34 feet under the mean pressure of one atmosphere, for the specific gravity of mercury is 13.6 times that of pure water, and as 30 inches is 2.5 feet, $13.6 \times 2.5 = 34.0$ feet. A water barometer is impracticable for use in measuring atmospheric pressures, but it is convenient to know its approximate height corresponding to a given height of the mercury barometer. The following table gives in the first column heights of the mercury barometer, in the second the corresponding pressures per square inch, in the third the pressures in atmospheres, and in the fourth the heights of the water barometer. This fourth column is computed by multiplying the numbers in the first column by 1.133, which is one-twelfth of 13.6, the specific gravity of mercury.

TABLE II. ATMOSPHERIC PRESSURE.

Mercury Barometer. Inches.	Pressure. Pounds per Square Inch.	Pressure. Atmospheres.	Water Barometer. Feet.	Elevations. Feet.	Boiling-point of Water (Fahrenheit).
31	15.2	1.03	35.1	− 895	213°.9
30	14.7	1.	34.0	0	212 .2
29	14.2	0.97	32.9	+ 925	210 .4
28	13.7	0.93	31.7	1880	208 .7
27	13.2	0.90	30.6	2870	206 .9
26	12.7	0.86	29.5	3900	205 .0
25	12.2	0.83	28.3	4970	203 .1
24	11.7	0.80	27.2	6085	201 .1
23	11.3	0.76	26.1	7240	199 .0
22	10.8	0.72	24.9	8455	196 .9
21	10.3	0.69	23.8	9720	194 .7
20	9.8	0.67	22.7	11050	192 .4

This table also gives in the fifth column values adapted from the vertical scale of altitudes used in barometric work, which show approximate vertical heights corresponding to

barometer readings, provided that the pressure at sea level is 30 inches.* In the last column are given the approximate boiling-points of water corresponding to the readings of the mercury barometer.

Prob. 5. What pressure in pounds per square inch exists at the base of a column of water 170 feet high? What pressure in atmospheres?

ARTICLE 5. COMPRESSIBILITY OF WATER.

The popular opinion that water is incompressible is not justified by experiments, which show in fact that it is more compressible than iron or even timber within the elastic limit. These experiments indicate that the amount of compression is directly proportional to the applied pressure, and that water is perfectly elastic, recovering its original form on the removal of the pressure. The amount of linear compression caused by a pressure of one atmosphere is, according to the measures of GRASSI, from 0.0000051 at 35° Fahrenheit to 0.000045 at 80° Fahrenheit.

Taking 0.00005 as a mean value of the linear compression per atmosphere, the coefficient of elasticity of water is

$$E = \frac{14.7}{0.00005} = 294\,000 \text{ pounds per square inch,}$$

which is only one-fifth of the coefficient of elasticity of timber, and less than one-eightieth that of wrought-iron.†

A column of water hence increases in density from the surface downward. If its weight at the surface be 62.5 pounds per cubic foot, at a depth of 34 feet a cubic foot will weigh

$$62.5\,(1 + 0.00005) = 62.503 \text{ pounds,}$$

* PLYMPTON, The Aneroid Barometer (New York, 1878).

† MERRIMAN's Mechanics of Materials (New York, 1885), p. 9.

and at a depth of 340 feet a cubic foot will weigh

$$62.5\,(1 + 0.0005) = 62.53 \text{ pounds.}$$

The variation in weight, due to compressibility, is hence too small to be regarded in hydrostatic computations.

Prob. 6. If w be the weight of water per cubic foot at the surface, show that the weight at a depth of d feet is $w\,(1 + 0.0000015\,d)$.

ARTICLE 6. THE ACCELERATION OF GRAVITY.

The symbol g is used in hydraulics to denote the acceleration of gravity; that is, the increase in velocity per second for a body falling freely in a vacuum at the surface of the earth. At the end of t seconds from the beginning of the fall, the velocity of the body is

$$V = gt.$$

The space, h, passed over in this time, is the product of the mean velocity, $\frac{1}{2}V$, and the number of seconds, t, or

$$h = \tfrac{1}{2}gt^2.$$

The relation between the velocity and the space is found by eliminating t from these two equations, and is

$$V = \sqrt{2gh}.$$

Hence the velocity of a body which has fallen freely through any height varies as the square root of that height. This equation may also be written in the form

$$h = \frac{V^2}{2g},$$

which shows that the height or space varies with the square of the velocity of the falling body.

The quantity 32.2 feet per second per second is an approximate value of g which is often used in hydraulic formulas. It is, however, well known that the force of gravity is not of constant intensity over the earth's surface, but is greater at the poles than at the equator, and also greater at the sea level than on high mountains. The following formula of PEIRCE,* which is partly theoretical and partly empirical, gives the value of g in feet for any latitude l, and any elevation e above the sea level, e being taken in feet:

$$g = 32.0894\,(1 + 0.0052375 \sin^2 l)(1 - 0.0000000957e):$$

and from this its value may be computed for any locality.

The greatest value of g is at the sea level at the pole, for which

$$l = 90°, \quad e = 0, \qquad \text{whence } g = 32.258.$$

The least value of g is on high mountains at the equator; for this there may be taken

$$l = 0°, \quad e = 10\,000 \text{ feet}, \quad \text{whence } g = 32.059.$$

Again, for the United States the practical limiting values are:

$$l = 49°, \quad e = 0, \qquad \text{whence } g = 32.186;$$
$$l = 25°, \quad e = 10\,000 \text{ feet}, \quad \text{whence } g = 32.089.$$

These results indicate that 32.2 feet is too large for a mean value of the acceleration.

In the numerical work of this book, the value of the acceleration is taken to be, unless otherwise stated,

$$g = 32.16 \text{ feet per second per second},$$

* SMITH'S Hydraulics, p. 19, where may be found a table giving values of $\sqrt{2g}$.

from which the frequently occurring quantity $\sqrt{2g}$ is found to be

$$\sqrt{2g} = 8.02.$$

If greater precision be required, which will rarely be the case, g can be computed from the formula for the particular latitude and elevation above sea level.

Prob. 7. Compute the value of g for the latitude 40° 36′, and the elevation 400 feet.

Prob. 8. What is the value of g if the unit of time be one minute? Ans. 115 776 feet per minute per minute.

ARTICLE 7. NUMERICAL COMPUTATIONS.

The numerical work of computation should not be carried to a greater degree of refinement than the data of the problem warrant. For instance, in questions relating to pressures, the data are uncertain in the third significant figure, and hence more figures than three or four in the final result must be delusive. Thus, let it be required to compute the number of pounds of water in a box containing 307.37 cubic feet. Taking the mean value 62.5 pounds as the weight of one cubic foot, the multiplication gives the result 19 210.625 pounds, but evidently the decimals here have no precision, since the last figure in 62.5 is not accurate, and is likely to be less than 5, depending upon the impurity of the water and its temperature. The proper answer to this problem is 19 200 pounds, or perhaps 19 210 pounds, and this is to be regarded as a probable average result rather than an exact definite quantity.

The use of logarithms is to be recommended in hydraulic computations, as thereby both mental labor and time are saved. Four-figure tables are sufficient for all common problems, and their use is particularly advantageous in cases where the data are not precise, as thus the number of significant figures in

results is kept at about three and statements implying great precision, when none really exists, are prevented. In some problems five-figure logarithms will be needed, but probably no hydraulic data are ever sufficiently exact to require the use of a seven-figure table. Six-figure logarithms should not be employed if others can be obtained, as their arrangement is not generally convenient for interpolation.

As this book is mainly intended for the use of students in technical schools, a word of advice directed especially to them may not be inappropriate. It will be necessary for students in order to gain a clear understanding of hydraulic science, or of any other engineering subject, to solve many numerical problems, and in this a neat and systematic method should be cultivated. The practice of performing computations on any loose scraps of paper that may happen to be at hand should not be followed, but the work should be done in a special book provided for that purpose, and be accompanied by such explanatory remarks as may seem necessary in order to render the solution clear. Such a note-book, written in ink, and containing the fully worked out solutions of the problems and examples given in these pages, will prove of great value to every student who makes it.

Prob. 9. Compute the weight of a column of water 1.1286 inches in diameter and 34.0 feet high at the temperature of 62° Fahrenheit.

Prob. 10. How many gallons of water are contained in a pipe 4 inches in diameter and 12 feet long? How many pounds?

CHAPTER II.

HYDROSTATICS.

ARTICLE 8. TRANSMISSION OF PRESSURES.

One of the most remarkable properties of water is its capacity of transmitting a pressure, applied at one point of the surface of a closed vessel, unchanged in intensity, in all directions, so that the effect of the applied pressure is to cause an equal force per square inch upon all parts of the enclosing surface. This is a consequence of the perfect fluidity of the water, by which its particles move freely over each other and thus transmit the applied pressure.

An experimental proof of this property is seen in the hydrostatic press, where the force applied to the small piston is exerted through the fluid and produces an equal unit-pressure at every point on the large piston. The applied force is here multiplied to any required extent, but the work performed by the large piston cannot exceed that imparted to the fluid by the small one. Let a and A be the areas of the small and large pistons, and p the pressure in pounds per square unit applied to a; then the total pressure on the small piston is pa, and that on the large piston is pA. Let the distances through which the pistons move at one stroke of the smaller be d and D. Then the imparted work is pad, and the performed work, neglecting hurtful resistances, is pAD. Consequently $ad = AD$, and since a is small as compared with A, the distance D must

be small compared with d. Here is found an illustration of the popular maxim that "What is gained in force is lost in velocity."

The pressure existing at any point within a body of water is exerted in all directions with equal intensity. This important property follows at once from that of the transmission of pressure, for this may be regarded as effected by the confined body of water acting as an elastic spring which presses outwards in all directions. Thus every particle of the water is in a state of stress, and reacts in all directions with equal intensity. And the same principle applies to a particle within a body of water whose surface is free, for the pressure which exists at any point due to the weight above it produces a state of stress among all the fluid particles.

Prob. 11. In a hydrostatic press a work of one-fourth a horse-power is applied to the small piston. The diameter of the large piston is 12 inches, and it moves half an inch per minute. Find the pressure per square inch in the fluid.

Ans. 1750 pounds.

ARTICLE 9. HEAD AND PRESSURE.

The free surface of water at rest is perpendicular to the direction of the force of gravity, and for bodies of water of small extent this surface may be regarded as a plane. Any depth below this plane is called "a head," or the head upon any point is its vertical depth below the level surface. Let h be the head and w the weight of a cubic unit of water; then at the depth h one horizontal square unit bears a pressure equal to the weight of a column of water whose height is h, and whose cross-section is one square unit, or wh. But the pressure at this point is exerted in all directions with equal intensity. The unit-pressure p at the depth h then is

$$p = wh; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and conversely the depth, or head, for a unit-pressure p is

$$h = \frac{p}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

If h be taken in feet and p in pounds per square foot, these formulas are

$$p = 62.5h,$$
$$h = 0.016p.$$

Hence pressure and head are mutually convertible, and in fact one is often used as synonymous with the other, although really each is proportional to the other. Any pressure p can be regarded as produced by a head h, which sometimes is called the "pressure head."

In numerical work p is usually taken in pounds per square inch, while h is expressed in feet. Thus, the pressure in pounds per square foot is $62.5h$, and the pressure in pounds per square inch is $\frac{1}{144}$ of this; or,

$$p = 0.434h,$$
$$h = 2.304p.$$

Stated in words these rules are:

1 foot head corresponds to 0.434 pounds per square inch;
1 pound per square inch corresponds to 2.304 feet head.

These values, be it remembered, depend upon the assumption that 62.5 pounds is the weight of a cubic foot of water, and hence are liable to variation in the third significant figure (Art. 3). The extent of these variations for fresh water may be judged by the following table, which gives multiples of the above values, and also the corresponding quantities when the cubic foot is taken as 62.3 pounds.

TABLE III. HEADS AND PRESSURES.

Head in Feet.	Pressure in Pounds per Square Inch.		Pressure in Pounds per Square Inch.	Head in Feet.	
	$w = 62.5$	$w = 62.3$		$w = 62.5$	$w = 62.3$
1	0.434	0.433	1	2.304	2.311
2	0.868	0.865	2	4.608	4.623
3	1.302	1.298	3	6.912	6.934
4	1.736	1.731	4	9.216	9.246
5	2.170	2.163	5	11.520	11.557
6	2.604	2.596	6	13.824	13.868
7	3.038	3.028	7	16.128	16.180
8	3.472	3.461	8	18.432	18.491
9	3.906	3.894	9	20.736	20.803
10	4.340	4.326	10	23.040	23.114

The atmospheric pressure, whose average value is 14.7 pounds per square inch, is transmitted through water, and is to be added to the pressure due to the head whenever it is necessary to regard the absolute pressure. This is important in some investigations on the pumping of water, and in a few other cases where a partial or complete vacuum is produced on one side of a body of water. For example, if the air be exhausted from a small globe, so that its tension is only 5 pounds per square inch, and it be submerged in water to a depth of 250 feet, the absolute pressure per square inch on the globe is

$$p = 0.434 \times 250 + 14.7 = 123.2 \text{ pounds,}$$

and the resultant effective pressure per square inch is

$$p' = 123.2 - 5.0 = 118.2 \text{ pounds.}$$

Unless otherwise stated, however, the atmospheric pressure need not be regarded, since under ordinary conditions it acts with equal intensity upon both sides of a submerged surface.

Prob. 12. What unit pressure corresponds to 230 feet head? What head in meters produces a pressure of 10 kilograms per square centimeter?

Prob. 13. A bottle contains air of 2 pounds per square inch tension, and its cork, 0.75 inches in diameter, can be forced in by a pressure of 100 pounds. How deep under water must it be sunk in order to force the cork? Ans. 492 feet.

ARTICLE 10. NORMAL PRESSURE.

The total normal pressure on any submerged surface may be found by the following theorem:

> The normal pressure is equal to the product of the weight of a cubic unit of water, the area of the surface, and the head on its centre of gravity.

To prove this let A be the area of the surface, and imagine it to be composed of elementary areas, a_1, a_2, a_3, etc., each of which is so small that the unit-pressure over it may be taken as uniform; let h_1, h_2, h_3, etc., be the heads on these elementary areas, and let w denote the weight of a cubic unit of water. The unit-pressures at the depths h_1, h_2, h_3, etc., are wh_1, wh_2, wh_3, etc. (Art. 9), and hence the normal pressures on the elementary areas a_1, a_2, a_3, etc., are wa_1h_1, wa_2h_2, wa_3h_3, etc. The total normal pressure P on the entire surface then is

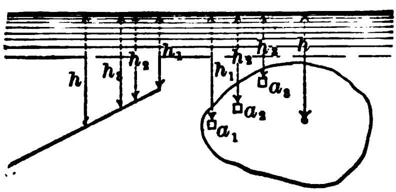

FIG. 2.

$$P = w(a_1h_1 + a_2h_2 + a_3h_3 + \text{etc.}).$$

Now let h be the head on the centre of gravity of the surface; then, from the definition of the centre of gravity,

$$a_1h_1 + a_2h_2 + a_3h_3 + \text{etc.} = Ah.$$

Therefore the normal pressure is

$$P = wAh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which proves the theorem as stated.

This rule applies to all surfaces, whether plane, curved, or warped, and however they be situated with reference to the water surface. Thus the total normal pressure upon the surface of a submerged cylinder remains the same whatever be its position, provided the depth of the centre of gravity of that surface be kept constant. It is best to take h in feet, A in square feet, and w as 62.5; then P will be in pounds. In case surfaces are given whose centres of gravity are difficult to determine, they should be divided into simpler surfaces, and then the total normal pressure is the sum of the normal pressures on the separate surfaces.

The normal pressure on the base of a vessel filled with water is equal to the weight of a cylinder of water whose base is the base of the vessel, and whose height is the depth of water, and only in the case of a vertical cylinder does this become equal to the weight of the water. Thus the pressure on the base of a vessel depends upon the depth of water and not upon the shape of the vessel. Also in the case of a dam, the depth of the water and not the size of the pond determines the amount of pressure.

The normal pressure on the interior surface of a sphere filled with water is greater than the weight of the water, for the weight acts only vertically, while the normal pressures are exerted in all directions. If d be the diameter of the sphere, formula (2) gives

$$P = w \cdot \pi d^2 \cdot \tfrac{1}{2}d = \tfrac{1}{2}w\pi d^3,$$

while the weight of water is

$$W = w \cdot \tfrac{1}{6}\pi d^3 = \tfrac{1}{6}w\pi d^3.$$

Hence the interior normal pressure is three times the weight of the water.

Prob. 14. A cone with altitude h and diameter of base d is filled with water. Find the normal pressure on the interior

surface (*a*) when it is held vertical with base downward; (*b*) when held horizontal.

Prob. 15. A board 2 feet wide at one end and 2 feet 6 inches at the other is 8 feet long. What is the normal pressure upon each of its sides when placed vertically in water with the narrow end in the surface?

ARTICLE 11. PRESSURE IN A GIVEN DIRECTION.

The pressure against a submerged plane surface in a given direction may be found by obtaining the normal pressure by Art. 10 and computing its component in the required direction, or by means of the following theorem:

> The horizontal pressure on any plane surface is equal to the normal pressure on its vertical projection; the vertical pressure is equal to the normal pressure on its horizontal projection; and the pressure in any direction is equal to the normal pressure on a projection perpendicular to that direction.

To prove this let A be the area of the given surface, represented by AA in Fig. 3, and P the normal pressure upon it, or $P = wAh$. Now let it be required to find the pressure P' in a direction making an angle θ with the normal to the given plane. Draw $A'A'$ perpendicular to the direction of P', and let A' be the area of the projection of A upon it. The value of P' then is

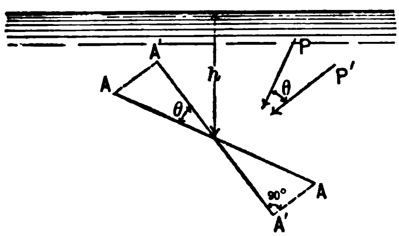

FIG. 3.

$$P' = P \cos \theta = wAh \cos \theta.$$

But $A \cos \theta$ is the value of A' by the construction. Hence

$$P' = wA'h, \qquad (3)$$

and the theorem is thus demonstrated.

This theorem does not in general apply to curved surfaces. But in cases where the head of water is so great that the pressure may be regarded as uniform it is also true for curved surfaces. For instance, consider a cylinder or sphere subjected on every elementary area to the unit-pressure p due to the high head h, and let it be required to find the pressure in the direction shown by q_1, q_2, and q_3 in Fig. 4. The pressures p_1, p_2, p_3, etc., on the elementary areas a_1, a_2, a_3, etc., are

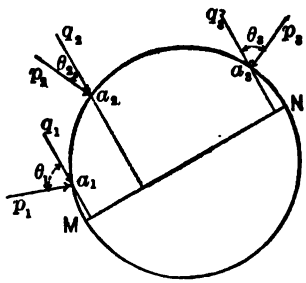

FIG. 4.

$$p_1 = pa_1, \quad p_2 = pa_2, \quad p_3 = pa_3, \quad \text{etc.},$$

and the components of these in the given direction are

$$q_1 = pa_1 \cos \theta_1, \quad q_2 = pa_2 \cos \theta_2, \quad q_3 = pa_3 \cos \theta_3, \quad \text{etc.},$$

whence the total pressure P' in the given direction is

$$P' = p(a_1 \cos \theta_1 + a_2 \cos \theta_2 + a_3 \cos \theta_3 + \text{etc.}).$$

But the quantity in the parenthesis is the projection of the surface on a plane perpendicular to the given direction, or MN. Hence

$$P' = p \times \text{area } MN,$$

which is the same rule as for plane surfaces.

For the case of a water-pipe let p be the interior pressure per square inch, and d its diameter in inches. Then for a length of one inch the force tending to rupture the pipe longitudinally is pd. This is resisted by the unit stress S in the walls of the pipe acting over the area $2t$, if t be the thickness. As these forces are equal,

$$2St = pd,$$

which is the fundamental equation for the discussion of the strength of water-pipes.

Prob. 16. The back of a dam has a slope of $1\frac{1}{2}$ to 1. Find the horizontal pressure per linear foot upon it, the water being 13 feet deep.

Prob. 17. What head of water will burst a pipe 24 inches in interior diameter and 0.75 inches thick, the tensile strength of the cast-iron being 20,000 pounds per square inch?

ARTICLE 12. CENTRE OF PRESSURE ON RECTANGLES.

The centre of pressure on a surface submerged in water is the point of application of the resultant of all the normal pressures upon it. The simplest and probably the most important case is the following:

> If a rectangle be placed with one end in the water surface, the centre of pressure is distant from that end two-thirds of its length.

This theorem will be proved by the help of the graphical illustration shown in Fig. 5. The rectangle, which in practice might be a board, is placed with its breadth perpendicular to the plane of the drawing, so that AB represents its edge. It is required to find the centre of pressure C. For any head h the unit-pressure is wh (Art. 9), and hence the unit-pressures on one side of AB may be graphically represented by arrows which form a triangle. Now if a force P equal to the total pressure is applied on the other side of the rectangle to balance these unit-pressures, it must be placed opposite to the centre of gravity of the triangle. Therefore

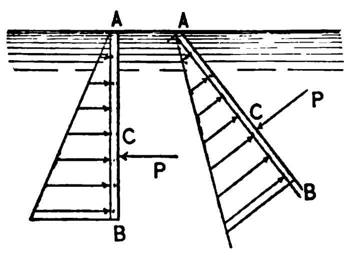

FIG. 5.

AC equals two-thirds of AB, and the rule is proved. The head on C is evidently also two-thirds of the head on B.

Another case is that shown in Fig. 6, where the rectangle, whose length is B_1B_2, is wholly immersed, the head on B_1 being h_1, and on B_2 being h_2. Let $AB_1 = b_1$, $AC = y$, and $AB_2 = b_2$. Now the normal pressure P_1 on AB_1 is applied at the distance $\frac{2}{3}b_1$ from A, and the normal pressure P_2 on AB_2 is applied at the distance $\frac{2}{3}b_2$ from A. The normal pressure P on B_1B_2 is the resultant of P_1 and P_2, or

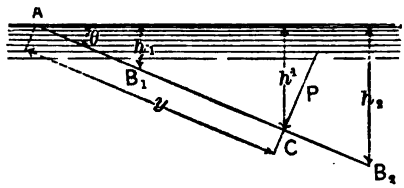

FIG. 6.

$$P = P_2 - P_1;$$

and also, by taking moments about A as a centre,

$$P \times y = P_2 \times \tfrac{2}{3}b_2 - P_1 \times \tfrac{2}{3}b_1.$$

Now, by Art. 10, the values of P_2 and P_1 are, for a rectangle one unit in breadth,

$$P_2 = w \times b_2 \times \tfrac{1}{2}h_2, \qquad P_1 = w \times b_1 \times \tfrac{1}{2}h_1;$$

hence
$$P = \tfrac{1}{2}w(b_2h_2 - b_1h_1);$$

and inserting these in the equation of moments, the value of y is

$$y = \frac{2}{3}\,\frac{b_2^2h_2 - b_1^2h_1}{b_2h_2 - b_1h_1}.$$

Now if θ be the angle of inclination of the plane to the water surface, $h_2 = b_2 \sin\theta$, and $h_1 = b_1 \sin\theta$. Accordingly, the expression becomes

$$y = \frac{2}{3}\,\frac{b_2^3 - b_1^3}{b_2^2 - b_1^2}.$$

Again, if h' be the head on the centre of pressure, $y = h'$ cosec θ, $b_2 = h_2$ cosec θ, and $b_1 = h_1$ cosec θ. These inserted in the last equation give

$$h' = \frac{2}{3} \frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}.$$

These formulas are very convenient for computation, as the squares and cubes may be taken from tables.

If h_1 equals h_2 the above formula becomes indeterminate, which is due to the existence of the common factor $h_2 - h_1$ in both numerator and denominator of the fraction; dividing out this common factor, it becomes

$$h' = \frac{2}{3} \frac{h_2^2 + h_2 h_1 + h_1^2}{h_2 + h_1},$$

from which, if $h_2 = h_1 = h$, there is found the result $h' = h$.

If $h_1 = 0$, or $b_1 = 0$, y becomes $\frac{2}{3} b_2$ and h' becomes $\frac{2}{3} h_2$, which proves again the special rule given at the beginning of this article.

Prob. 18. A rectangle 4 feet long is immersed in water with its ends parallel to the surface, the head on one end being 7 feet and that on the other 9 feet. Find the head on the centre of pressure, and also the value of P.

ARTICLE 13. GENERAL RULE FOR CENTRE OF PRESSURE.

For any plane surface submerged in a liquid, the centre of pressure may be found by the following rule:

> Find the moment of inertia of the surface and its statical moment, both with reference to an axis situated at the intersection of the plane of the surface with the water level. Divide the former by the latter, and the quotient is the perpendicular distance from that axis to the centre of pressure.

The demonstration is analogous to that in the last article. Let, in Fig. 6, B_1B_2 be the trace of the plane surface, which itself is perpendicular to the plane of the drawing, and C be the centre of pressure, at a distance y from A where the plane of the surface intersects the water level. Let a_1, a_2, a_3, etc., be elementary areas of the surface, and h_1, h_2, h_3, etc., the heads upon them, which produce the normal elementary pressures, wa_1h_1, wa_2h_2, wa_3h_3, etc. Let y_1, y_2, y_3, etc., be the distances from A to these elementary areas. Then taking the point A as a centre of moments, the definition of centre of pressure gives the equation

$$(wa_1h_1 + wa_2h_2 + wa_3h_3 + \text{etc.})\,y = wa_1h_1y_1 + wa_2h_2y_2 + wa_3h_3y_3 + \text{etc.}$$

Now let θ be the angle of inclination of the surface to the water level; then $h_1 = y_1 \sin\theta$, $h_2 = y_2 \sin\theta$, $h_3 = y_3 \sin\theta$, etc. Hence, inserting these values, the expression for y is

$$y = \frac{a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + \text{etc.}}{a_1y_1 + a_2y_2 + a_3y_3 + \text{etc.}}.$$

The numerator of this fraction is the sum of the products obtained by multiplying each element of the surface by the square of its distance from the axis, which is called the moment of inertia of the surface. And the denominator is the sum of the products of each element of the surface by its distance from the axis, which is called the statical moment of the surface. Therefore

$$y = \frac{\text{moment of inertia}}{\text{statical moment}} = \frac{I'}{S}. \quad . \quad . \quad . \quad . \quad (4)$$

is the general rule for finding centres of pressure for plane surfaces.

The statical moment of a surface is simply its area multiplied by the distance of its centre of gravity from the given

axis, as is evident from the definition of centre of gravity. The moments of inertia of plane surfaces with reference to an axis through the centre of gravity are deduced in works on theoretical mechanics; a few values are:

For a rectangle of breadth b and depth d, $I = \frac{bd^3}{12}$;

For a triangle with base b and altitude d, $I = \frac{bd^3}{36}$;

For a circle with diameter d, $I = \frac{\pi d^4}{64}$.

To find from these the moment of inertia with reference to a parallel axis, the well-known formula $I' = I + Ak^2$ is to be used, where A is the area of the surface and k the distance from the given axis to the centre of gravity of the surface, and I' the moment of inertia required.

For example, let it be required to find the centre of pressure of a circle which is submerged with one edge in the water surface. The area of the circle is $\frac{1}{4}\pi d^2$, and its statical moment with reference to the upper edge is $\frac{1}{4}\pi d^2 \times \frac{1}{2}d$. Then from (4),

$$y = \frac{\frac{\pi d^4}{64} + \frac{\pi d^2}{4} \cdot \frac{d^2}{4}}{\frac{\pi d^2}{4} \cdot \frac{d}{2}} = \frac{5}{8}d;$$

hence the centre of pressure of a circle with one edge in the water surface is at $\frac{1}{8}d$ below the centre. Again, for a triangle submerged with its vertex in the water surface,

$$y = \frac{\frac{bd^3}{36} + \frac{bd}{2} \cdot \frac{4d^2}{9}}{\frac{bd}{2} \cdot \frac{2d}{3}} = \frac{3}{4}d.$$

Prob. 19. Find the centre of pressure of the triangle in Fig. 9 when it is inverted so that its base is in the surface.

Prob. 20. Find the centre of pressure of a circle when vertically submerged in water so that the head on its centre is equal to two diameters of the circle. Ans. 2.03*d*.

ARTICLE 14. PRESSURES ON OPPOSITE SIDES OF A PLANE.

In the case of an immersed plane the water presses equally upon both sides so that no disturbance of the equilibrium results from the pressure. But in case the water is at different levels on opposite sides of the surface the opposing pressures are unequal. For example, the cross-section of a self-acting tide-gate, built to drain a salt marsh, is shown in Fig. 7. On the ocean side there is a head of h_1 above the sill, which gives for every linear foot of the gate the pressure

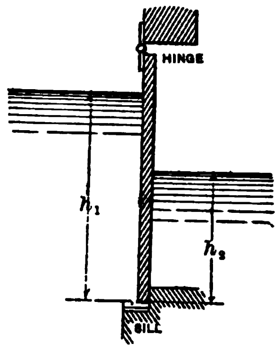

FIG. 7.

$$P_1 = w \times h_1 \times \tfrac{1}{2}h_1 = \tfrac{1}{2}wh_1^2$$

which is applied at the distance $\frac{1}{3}h_1$ above the sill. On the other side the head on the sill is h_2, which gives the pressure

$$P_2 = \tfrac{1}{2}wh_2^2,$$

whose centre of pressure is at $\frac{1}{3}h_2$ above the sill. The resultant pressure P is

$$P = P_1 - P_2 = \tfrac{1}{2}w(h_1^2 - h_2^2);$$

and if z be the distance of the point of application of P above the sill, the equation of moments is

$$(P_1 - P_2)z = P_1 \times \tfrac{1}{3}h_1 - P_2 \times \tfrac{1}{3}h_2,$$

from which z can be computed.

The action of the gate in resisting the water pressure is like that of a beam under its load, the two points of support being at the sill and the hinge. If h be height of the gate, the reaction at the hinge is,

$$R = (P_1 - P_2)\frac{z}{h} = \frac{P_1h_1 - P_2h_2}{3h},$$

and this has its greatest value when h_1 becomes equal to h, and h_2 is zero. In the case of the vertical gate of a canal lock, which swings horizontally like a door, a similar problem arises and a similar conclusion results.

Prob. 21. If the head on one side of a tide-gate is 7 feet and on the other 4 feet, find the resultant pressure and its point of application above the sill.

Ans. 1031 pounds per linear foot, at 2.82 feet above sill.

Article 15. Masonry Dams.

The preceding articles show that the pressure on the back of a masonry dam is normal to that surface at every point. If the back be a plane surface the resultant pressure is normal to the plane, and its point of application is at two-thirds of the length from the water level. Thus in Fig. 8, AC is two-thirds of AB. If h be the head of water above the base of the dam, and θ be the angle of inclination of the plane of the back to the vertical, the normal pressure per linear foot of the dam is, from Art. 10,

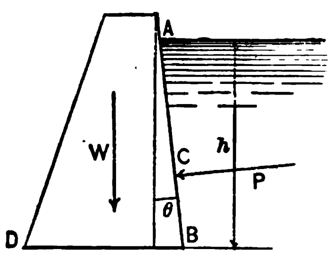

Fig. 8.

$$P = w \times h \sec\theta \times \tfrac{1}{2}h = \tfrac{1}{2}wh^2 \sec\theta,$$

which shows that the total pressure against the dam varies as the square of its height. The horizontal component of this

pressure is $\frac{1}{2}wh^2$, which is the same as the normal pressure against a wall whose back is vertical.

It is not the place here to enter into the discussion of the subject of the design of masonry dams, but the two ways in which they are liable to fail may be noted. The first is that of sliding along a horizontal joint, as BD; here the horizontal component of the thrust overcomes the resisting force of friction acting along the joint. If W be the weight of masonry above the joint, and f the coefficient of friction, the resisting friction is fW, and the dam will slide if the horizontal component of the pressure is equal to or greater than this. The condition for failure by sliding then is

$$\tfrac{1}{2}wh^2 = fW.$$

The second method of failure is that of rotating around the toe D: this occurs when the moment of P equals the moment of W with reference to that point; or if l and m be the lever-arms dropped from D upon the directions of P and W, the condition for failure by rotation is

$$Pl = Wm.$$

In practice the joints are so built as to give full security against sliding, so that the usual method of failure is by rotation.

As an example of the application of these principles consider a rectangular vertical masonry dam which weighs 140 pounds per cubic foot, and which is 4 feet wide. First, let it be required to find the height for which it would fail by sliding, the coefficient of friction being 0.75. The horizontal water pressure is $\frac{1}{2} \times 62.5 \times h^2$, and the resisting friction is $0.75 \times 140 \times 4 \times h$. Placing these equal, there is found $h =$ 13.4 feet. Secondly, to find the height for which failure will occur by rotation, the equation of moments is stated with ref-

erence to the front lower edge, the lever-arm of the pressure being $\frac{1}{3}h$, and that of the wall 2 feet. Hence

$$\tfrac{1}{2} \times 62.5 \times h^2 \times \tfrac{1}{3}h = 140 \times 4 \times h \times 2,$$

from which there is found $h = 10.4$ feet.

Prob. 22. A dam whose cross-section is a triangle has a vertical back, is 3 feet wide at the base, and 15 feet high. Find the height to which the water may rise behind it in order to cause failure (*a*) by sliding, and (*b*) by rotation, using 0.75 for the coefficient of friction and 140 pounds per cubic foot for the weight of the masonry.

ARTICLE 16. LOSS OF WEIGHT IN WATER.

It is a familiar fact that bodies submerged in water lose part of their weight: a man can carry under water a large stone which would be difficult to lift in air; timber when submerged has a negative weight or tends to rise to the surface. The following is the law of loss: *

> The weight of a body submerged in water is less than its weight in air by the weight of a volume of water equal to that of the body.

To demonstrate this, consider that the submerged body is acted upon by the water pressure in all directions, and that the horizontal components of these pressures must balance. Any vertical elementary prism is subjected to an upward pressure upon its base which is greater than the downward pressure upon its top, since these pressures are due to the heads. Let h_1 be the head on the top of the elementary prism and h_2 that on its base, and a the cross-section of the prism; then the downward pressure is wah_1 and the upward pressure is wah_2. The difference of these, $wa(h_2 - h_1)$ is the resultant upward water pressure, and this is equal to the weight of a column of water whose cross-section is a and whose height

* Discovered by ARCHIMEDES, about 250 B.C.

is that of the elementary prism. Extending this to all the elementary prisms which make up the body, it is seen that the upward water pressure diminishes its weight by the weight of a volume of water equal to that of the body.

It is important to regard this loss of weight in constructions under water. If, for example, a dam of loose stones allows the water to percolate through it, its weight per cubic foot is less than its weight in air, so that it can be more easily moved by horizontal forces. As stone weighs about 150 pounds per cubic foot in air, its weight in water is only about $150 - 62 = 88$ pounds.

Prob. 23. A bar of iron one square inch in cross-section and one yard long weighs 10 pounds in air. What is its weight in water?

ARTICLE 17. DEPTH OF FLOTATION.

When a body floats upon water it is sustained by an upward pressure of the water equal to its own weight, and this pressure is the same as the weight of the volume of water displaced by the body. Let W' be the weight of the floating body in air, and W be the weight of the displaced water; then

$$W' = W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now let z be the depth of flotation of the body; then to find its value for any particular case W' is to be expressed in terms of the linear dimensions of the body, and W in terms of the depth of flotation z.

For example, a cone which weighs w' pounds per cubic foot floats with its base downward as represented in Fig. 9, its altitude being d and the radius of its base b. The weight of the floating cone is

$$W' = w' \cdot \pi b^2 \cdot \tfrac{1}{3}d,$$

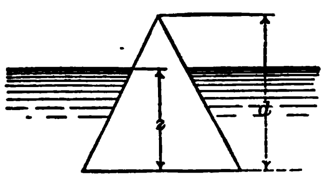

FIG. 9.

and the weight of the displaced water is that of a frustum of the altitude z, or

$$W = w\left(\pi b^2 \cdot \frac{1}{3} d - \pi \left(\frac{d-z}{d} b\right)^2 \frac{d-z}{3}\right).$$

Equating these values and solving for z gives the result

$$z = d - d\left(1 - \frac{w'}{w}\right)^{\frac{1}{3}},$$

which is the depth of flotation. If $w' = w$, the cone has the same density as water, and $z = d$; if $w' = 0$, the cone has no weight, and $z = 0$.

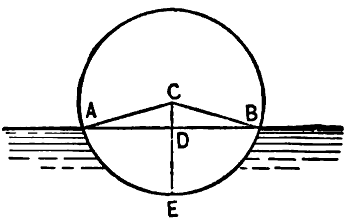

FIG. 10.

To find the depth of flotation for a cylinder lying horizontally, let w' be its weight per cubic foot, and r the radius of its cross-section. The depth of flotation is DE (Fig. 10), or if θ be the angle ACE,

$$z = r(1 - \cos\theta).$$

The weight of the cylinder for one unit of length is

$$W' = w' \cdot \pi r^2,$$

and that of the displaced water is

$$W = w(r^2 \text{ arc } \theta - r^2 \sin\theta \cos\theta).$$

Equating the values of W and W', and substituting for $\sin\theta\cos\theta$ its equivalent $\frac{1}{2}\sin 2\theta$, there results

$$2 \text{ arc } \theta - \sin 2\theta = 2\pi\frac{w'}{w}.$$

From this equation θ is to be found by trial for any particular

case, and then z is known. For example, if $w' = 26.5$ pounds per cubic foot, and $r = 12$ inches,

$$2 \text{ arc } \theta - \sin 2\theta - 2.664 = 0.$$

To solve this equation, assume values for θ, until finally one is found that satisfies it; thus:

$$\text{For } \theta = 83^\circ, \quad 2.897 - 0.242 - 2.664 = -0.009;$$
$$\text{For } \theta = 83\tfrac{1}{4}, \quad 2.906 - 0.234 - 2.664 = +0.008.$$

Therefore θ lies between 83° and 83° 15′, and is probably about 83° 8′. Hence the depth of flotation is $z = 12(1 - 0.120) =$ 10.6 inches.

Prob. 24. Show that the depth of flotation for a sphere whose radius is r is the real root of the cubic equation $z^3 - 3rz^2 + 4r^3 \dfrac{w'}{w} = 0$.

Prob. 25. A rectangular wooden box 4.5 feet long, 3 feet wide, and 2.5 feet deep, inside dimensions, is made of timber $1\frac{1}{2}$ inches thick, which weighs 3 pounds per foot board measure. How much water will it draw when a weight of 200 pounds is placed in it and the cover nailed on? Ans. 0.46 feet.

ARTICLE 18. STABILITY OF FLOTATION.

The equilibrium of a floating body is stable when it returns to its primitive position after having been slightly moved therefrom by extraneous forces, it is indifferent when it floats in any position, and it is unstable when the slightest force causes it to leave its position of flotation. For instance, a short cylinder with its axis vertical floats in stable equilibrium, but a long cylinder in this position is unstable, and a slight force causes it to fall over and float with its axis horizontal in indifferent equilibrium.

The stability depends in any case upon the relative position of the centre of gravity of the body and its centre of buoyancy, the latter being the centre of gravity of the displaced water. Thus in Fig. 11 let G be the centre of gravity of the body and C that of the centre of buoyancy when in an upright position. Now if an extraneous force causes the body to tip into the position shown, the centre of gravity remains at G, but the centre of buoyancy moves to D. In this new position of the body it is acted upon by the forces W' and W, whose lines of direction pass through G and D. W' is the weight of the body and W the weight of the displaced water; and as these are equal, they form a couple which tends either to restore the body to the upright position or to cause it to deviate farther from that position. Let the vertical through D be produced to meet the centre line CG in M. If M is above G the equilibrium is stable, as the forces W and W' tend to restore it to its primitive position; if M coincides with G the equilibrium is indifferent; and if M be below G the equilibrium is unstable.

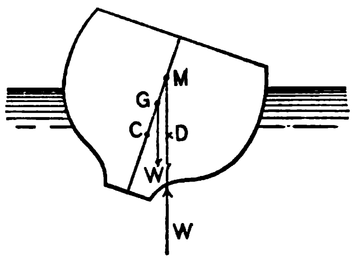

FIG. 11.

The point M is often called the 'metacentre,' and the theorem may be stated that the equilibrium is stable, indifferent, or unstable according as the metacentre is above, coincident with, or below the centre of gravity of the body. The measure of the stability of a floating body is the moment of the couple formed by the forces W and W'. But the line GM is proportional to the lever arm of the couple, and hence the quantity $W \times \overline{GM}$ may be taken as a measure of the stability. The stability, therefore, increases with the weight of the body, and with the distance of the metacentre above the centre of gravity.

To ensure a high degree of stability the centre of gravity should be as low as possible.

The only important applications of these principles are in connection with the subject of naval architecture, and in general the resulting investigations are of a complex character, which can only be solved by approximate tentative methods. REED'S Treatise on the Stability of Ships (London, 1885) is a large volume entirely devoted to this topic.

Prob. 26. If θ be the angle of inclination to the vertical, c the distance between the metacentre and centre of gravity, show that the stability of flotation can be measured by the quantity $Wc \sin \theta$.

CHAPTER III.

THEORETICAL HYDRAULICS.

ARTICLE 19. VELOCITY AND DISCHARGE.

If a vessel or pipe be constantly full of water, all the particles of which move with the same uniform velocity v, and if a be the area of its cross-section, the quantity of water which passes any section per second is equal to the volume of a prism whose base is a and whose length is v, or

$$q = av. \qquad \qquad (6)$$

If, now, the vessel varies in cross-section, one area being a, another a_1, and a third a_2, the same quantity of water passes each section per second if the vessel be kept constantly full; hence if v, v_1, and v_2 be the respective velocities,

$$q = av = a_1v_1 = a_2v_2.$$

The velocities of flow in different sections of a pipe or vessel which is maintained constantly full hence vary inversely as the areas of the cross-sections.

In case the particles or filaments move with different velocities in different parts of the section, the quantity may be still

expressed by $q = av$, provided that v signifies the mean velocity of the flow; or

$$v = \frac{q}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)'$$

may be regarded as a definition of the term mean velocity.

The word discharge will be used to denote the quantity of water flowing per second from a pipe or orifice, and the letter Q will designate the theoretic discharge, that is, the discharge as computed by the methods of this chapter, where resistances or losses due to friction, contraction, and other causes are not considered. The letter V will designate the theoretic velocity, so that if a be the area of an orifice, or the cross-section of a jet,

$$Q = aV$$

is the formula for the theoretic discharge. In the case of flow from a simple orifice the area a is found by the measurement of its dimensions, so that the problem of finding Q is reduced to that of determining V.

Prob. 27. A pipe constantly filled with water discharges 0.43 cubic feet per second. Compute the mean velocity of flow if the pipe is 3 inches in diameter; also if it is 6 inches in diameter.

ARTICLE 20. VELOCITY OF FLOW FROM ORIFICES.

If an orifice be opened, either in the base or side of a vessel containing water, it flows out with a velocity which is greater for high heads of water than for low heads. The theoretic velocity of flow is given by the theorem discovered by TORRICELLI.*

> The theoretic velocity of flow at the orifice is the same as that acquired by a body falling freely in a vacuum through a height equal to the head of water on the orifice.

* Del moto dei gravi (Firenz, 1644).

The proof of this rests partly on observation. Thus if a vessel be arranged, as in Fig. 12, so that a jet of water from an orifice is directed vertically upward, it is known that it never attains to the height of the level of the water in the vessel, although under favorable conditions it nearly reaches that level. It may hence be inferred that the jet would actually rise to that height were it not for the resistance of the air and the friction of the edges of the orifice. Now, since the velocity of impulse required to raise a body vertically to a certain height is the same as that acquired by it in falling from rest through that height, it is regarded as established that the velocity at the orifice is as stated in the theorem.

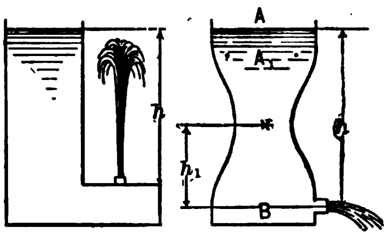

FIG. 12.

The following proof rests on the law of conservation of energy. Let, as in the second diagram of Fig. 12, the water surface in a vessel be at A at the beginning of a second and at A_1 at the end of the second. Let W be the weight of water between the planes A and A_1, which is evidently the same as that which flows from the orifice during the second. Let W_1 be the weight of water between the planes A_1 and B, and h_1 the height of its centre of gravity above the orifice. Let h be the height of A above the orifice, and δh the distance between A and A_1. Then at the beginning of the second the water in the vessel has the energy $W_1h_1 + W(h - \frac{1}{2}\delta h)$. If V be the velocity of flow through the orifice, the same water at the end of the second has the energy $W_1h_1 + W\frac{V^2}{2g}$. By the law of conservation these are equal, if no energy has been dissipated in friction or in other ways; thus,

$$h - \tfrac{1}{2}\delta h = \frac{V^2}{2g}.$$

Now if δh be small compared with h, which will be the case when A is large compared with the area of the orifice, this gives

$$V = \sqrt{2gh},$$

which is the same as for a body falling freely through the height h (Art. 6).

The theoretic velocity of flow from any orifice, whether its plane be horizontal, vertical, or inclined, is thus given by

$$V = \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

provided the orifice be small compared with the section of the reservoir. The theoretic height to which the jet will rise is

$$h = \frac{V^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)'$$

The first of these formulas states the velocity due to a given head, and the second states the head which would generate a given velocity. The term "velocity head" will be generally used to designate the expression $\frac{V^2}{2g}$, meaning thereby that its value is the head which can produce the velocity V.

Using for g the mean value 32.16 feet per second per second, these formulas become

$$V = 8.02\sqrt{h}, \qquad h = 0.01555V^2,$$

from which the following tables have been computed. These are mainly intended to impress upon the student the fact that small heads produce rapid velocities, but they may also prove serviceable for use in approximate computations. The last columns of the tables give multiples of the numbers 8.02 and 0.01555.

TABLE IV. THEORETIC VELOCITIES.

Head. Feet.	Velocity. Feet per Second.	Head. Feet.	Velocity. Feet per Second.	Multiples of 8.01997.	
0.001	0.254	1	8.02	1	8.02
.002	.358	2	11.33	2	16.04
.003	.439	3	13.89	3	24.06
.004	.507	4	16.04	4	32.08
.005	.567	5	17.93	5	40.10
.006	.621	6	19.64	6	48.12
.007	.671	7	21.22	7	56.14
.008	.717	8	22.68	8	64.16
.009	.761	9	24.06	9	72.18

TABLE V. VELOCITY HEADS.

Velocity. Feet per Second.	Head. Feet.	Velocity. Feet per Second.	Head. Feet.	Multiples of 0.015547.	
1	0.016	10	1.55	1	0.01555
2	.062	20	6.22	2	.03109
3	.140	30	13.99	3	.04664
4	.249	40	24.88	4	.06219
5	.389	50	38.86	5	.07774
6	.560	60	55.97	6	.09328
7	.762	70	76.19	7	.10883
8	.995	80	99.51	8	.12438
9	1.260	90	125.95	9	.13992

Prob. 28. Find the theoretic velocity of flow from an orifice under a head of 6 inches. Find the velocity-head of a stream 0.1 feet in diameter which discharges 2.5 cubic feet per minute.

Ans. $V = 5.67$ feet per second, $H = 0.44$ feet.

ARTICLE 21. HORIZONTAL ORIFICES.

Let a be the area of an orifice whose plane is horizontal, h the head of water upon it, and Q the quantity of water discharged per second. The theoretic discharge is, from the principles of the preceding articles,

$$Q = a\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided that the area of the orifice be small compared with the cross-section of the vessel. If a is in square feet and h in feet, Q will be expressed in cubic feet per second. It will be seen in the next chapter that various circumstances materially modify in practice the results as obtained from this formula.

The discharge from a horizontal orifice is, like the velocity, proportional to the square root of the head. Thus with the same orifice to double the discharge requires the head to be increased fourfold. The head which will produce a given discharge is

$$h = \frac{Q^2}{2ga^2},$$

whence the head varies inversely as the square of the area of the orifice.

Horizontal orifices are but little used, as in practice it is found more convenient to arrange an opening in the side of a vessel than in the base. The above formula applies approximately to a vertical orifice if h be taken as the head on its centre of gravity.

The discharge is theoretically independent of the shape of the orifice, so that orifices of different forms with equal areas give the same value of Q. For a circle whose diameter is d,

$$Q = \tfrac{1}{4}\pi d^2 \sqrt{2gh}.$$

For a rectangle whose sides are b and d,

$$Q = bd\sqrt{2gh};$$

and similarly for other forms a is to be inserted in terms of the linear dimensions, which must be numerically expressed in the same unit as g.

Prob. 29 Compute the theoretic discharge from an orifice one inch in diameter under a head of 1.5 feet.

ARTICLE 22. RECTANGULAR VERTICAL ORIFICES.

If the size of an orifice in the side of a vessel be small compared with the head, then the mean theoretic velocity of the outflowing water may be taken as $\sqrt{2gh}$, where h is the head on the centre of the orifice, and consequently the theoretic discharge is aV or $a\sqrt{2gh}$. Strictly, however, the head, and hence the velocity, is different in different parts of an orifice whose plane is vertical.

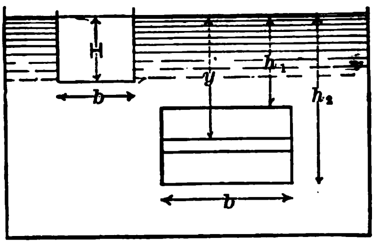

FIG. 13.

A rectangular orifice with two edges parallel to the water surface is the most important case. Let b be its breadth, h_1 the head of water on its upper edge, and h_2 the head on its lower edge, so that $h_2 - h_1$ is its depth. Let any elementary strip whose area is $b\,.\,\delta y$ be drawn at a depth y below the water level. The velocity of flow through this elementary strip is, as shown in Art. 20,

$$V = \sqrt{2gy},$$

and the discharge per second through it is

$$\delta Q = b\delta y\sqrt{2gy}.$$

The total discharge through the orifice is obtained by integrating this expression between the limits h_1 and h_2, which gives

$$Q = \tfrac{2}{3}b\sqrt{2g}(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

In case the top edge of the orifice is at or above the level of the water, $h_1 = 0$, and then if the head h_2 be denoted by H, the discharge is

$$Q = \tfrac{2}{3}b\sqrt{2g}H^{\frac{3}{2}} = \tfrac{2}{3}bH\sqrt{2gH} = \tfrac{2}{3}a\sqrt{2gH}, \quad . \quad . \qquad (9)'$$

which is the basis of all formulas for weir measurement.

To ascertain the error caused by using the formula (8) instead of (9) for a rectangular lateral orifice, let h be the head on its centre of gravity, and d be its vertical depth, $h_2 - h_1$. Then from (8)

$$Q = bd\sqrt{2gh}.$$

Now in (9) let $h_2 = h + \frac{1}{2}d$, and $h_1 = h - \frac{1}{2}d$; then developing by the binomial formula,

$$h_2^{\frac{3}{2}} = h^{\frac{3}{2}}\left(1 + \frac{3}{4}\frac{d}{h} + \frac{3}{32}\frac{d^2}{h^2} - \frac{1}{128}\frac{d^3}{h^3} + \frac{3}{2048}\frac{d^4}{h^4} - \frac{3}{8192}\frac{d^5}{h^5} + \text{etc.}\right);$$

$$h_1^{\frac{3}{2}} = h^{\frac{3}{2}}\left(1 - \frac{3}{4}\frac{d}{h} + \frac{3}{32}\frac{d^2}{h^2} + \frac{1}{128}\frac{d^3}{h^3} + \frac{3}{2048}\frac{d^4}{h^4} + \frac{3}{8192}\frac{d^5}{h^5} + \text{etc.}\right);$$

and (9) becomes

$$Q = bd\sqrt{2gh}\left(1 - \frac{1}{96}\frac{d^2}{h^2} - \frac{1}{2048}\frac{d^4}{h^4} - \frac{1}{21845}\frac{d^6}{h^6} - \text{etc.}\right).$$

Therefore the discharge obtained by using (8) is always too great. The true theoretic discharge, from the formula just deduced, is:

If $h = d$,	$Q = 0.989\, bd\sqrt{2gh}$;
If $h = 2d$,	$Q = 0.997\, bd\sqrt{2gh}$;
If $h = 3d$,	$Q = 0.999\, bd\sqrt{2gh}$.

The error of the formula $Q = bd\sqrt{2gh}$ is thus seen to be 1.1 per cent when $h = d$, only 0.3 per cent when $h = 2d$, and only about 0.1 per cent when $h = 3d$. Accordingly, if the head on the centre of the orifice is greater than two or three times the vertical depth of the orifice, the approximate formula (8) is generally used instead of the exact formula (9), since the slight error thus introduced is of no practical importance.

Prob. 30. Compute the theoretic discharge from a rectangular orifice 0.5 feet wide and 0.25 feet high when the head on the top of the orifice is 0.375 feet.

Ans. $Q = 0.707$ cubic feet per second.

ARTICLE 23. TRIANGULAR VERTICAL ORIFICES.

Triangular vertical orifices arc sometimes used for the measurement of water, the arrangement being as shown in Fig. 14. Let b be the width of the orifice at the water level, and H the head of water on the vertex. Let an elementary strip whose depth is dy be drawn at a distance y below the water level. From similar triangles the length of this strip is $\frac{b}{H}(H-y)$, and the elementary discharge then is

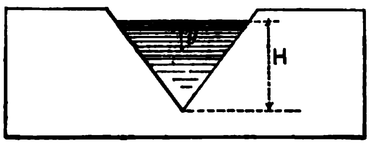

FIG. 14.

$$\delta Q = \frac{b}{H}(H-y)\delta y\sqrt{2gy} = \frac{b}{H}\sqrt{2g}(Hy^{\frac{1}{2}} - y^{\frac{3}{2}})\delta y.$$

The integration of this between the limits 0 and H gives

$$Q = \tfrac{4}{15}b\sqrt{2g}H^{\frac{3}{2}} = \tfrac{4}{15}bH\sqrt{2gH}.$$

If the sides of the triangle are equally inclined to the vertical, as should be the case in practice, and if this angle be α, b may be expressed in terms of α and H, so that the equation becomes

$$Q = \tfrac{8}{15}\tan\alpha \,.\, H^2\sqrt{2gH} = \tfrac{8}{15}\tan\alpha \,.\, \sqrt{2g} \,.\, H^{\frac{5}{2}}.$$

The discharge is thus equal to a constant multiplied by the $2\frac{1}{2}$ power of the measured depth.

If the orifice be a trapezoid whose upper base is b, lower base b', and altitude d, the discharge is found by integrating the above differential expression between the limits 0 and d, and then substituting for H its value in terms of d, b, and b', namely, $H = \frac{db}{b-b'}$. The theoretic discharge then is

$$Q = \tfrac{2}{3}bd\sqrt{2gd}\left(\frac{2}{5} + \frac{3}{5}\frac{b'}{b}\right).$$

If in this b' equals b it becomes the same as the formula for a rectangular orifice, while if b' equals 0 it gives the same result as found above for the triangle.

Prob. 31. Prove that the theoretic discharge from a lateral triangular orifice whose base is horizontal and whose vertex is in the water level is $Q = \frac{2}{5}bd\sqrt{2gd}$, where b is the base and d is the altitude.

ARTICLE 24. CIRCULAR VERTICAL ORIFICES.

To determine the theoretic discharge through a circular orifice whose plane is vertical, let h be the head on its centre, and r its radius. Let an elementary strip be drawn at a distance y above the centre; the length of this is $2\sqrt{r^2 - y^2}$, its area is $2\delta y\sqrt{r^2 - y^2}$, and the head upon it is $h - y$. Then the theoretic discharge through this strip is

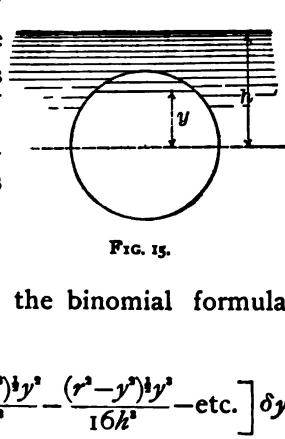

FIG. 15.

$$\delta Q = 2\delta y \sqrt{r^2 - y^2}\sqrt{2g(h - y)}.$$

To integrate this expand $(h - y)^{\frac{1}{2}}$ by the binomial formula. Then it may be written

$$\delta Q = 2\sqrt{2gh}\left[(r^2-y^2)^{\frac{1}{2}} - \frac{(r^2-y^2)^{\frac{1}{2}}y}{2h} - \frac{(r^2-y^2)^{\frac{1}{2}}y^2}{8h^2} - \frac{(r^2-y^2)^{\frac{1}{2}}y^3}{16h^3} - \text{etc.}\right]\delta y.$$

Each term of this expression is now integrable, and taking the limits of y as $+r$ and $-r$ the entire circle is covered, and

$$Q = \pi r^2\sqrt{2gh}\left(1 - \frac{1}{32}\frac{r^2}{h^2} - \frac{5}{1024}\frac{r^4}{h^4} - \frac{105}{65536}\frac{r^6}{h^6} - \text{etc.}\right), \quad (10)$$

which gives the theoretic discharge per second for any values of r and h.

The approximate formula (8) applied to this case gives for the discharge $\pi r^2 \sqrt{2gh}$, which is always greater than the true discharge; thus from (10),

$$\text{If} \quad h = 2r, \quad Q = 0.992\,\pi r^2 \sqrt{2gh};$$
$$\text{If} \quad h = 3r, \quad Q = 0.996\,\pi r^2 \sqrt{2gh};$$
$$\text{If} \quad h = 4r, \quad Q = 0.998\,\pi r^2 \sqrt{2gh}.$$

Hence the error in the use of (8) is only 0.4 per cent when $h = 3r$, and only 0.2 per cent when $h = 4r$. In general the approximate formula may be used whenever the head on the centre of the circle is greater than four or five times its radius.

Prob. 32. Compute the theoretic discharge from a circle of one inch diameter when the head on its centre is 0.5 feet.

ARTICLE 25. INFLUENCE OF VELOCITY OF APPROACH.

Thus far, in the determination of the theoretic velocity and discharge from an orifice, the head has been regarded as constant. But the head can only be maintained constant by an inflow of water, and this modifies the theoretic velocity. Let a be the area of the orifice, and A that of the horizontal cross-section of the reservoir; let V be the theoretic velocity of flow through a, and v the vertical velocity of inflow through the section A. The energy of W pounds of water as it flows from the orifice is $W\frac{V^2}{2g}$, and this is equal to the energy Wh stored up in the fall plus the energy $W\frac{v^2}{2g}$ of the inflowing water, or

$$W\frac{V^2}{2g} = Wh + W\frac{v^2}{2g}.$$

Now the quantity of water which flows through the section a

in a unit of time is the same as that passing through the area A in the same time, or (Art. 19)

$$aV = Av, \quad \text{whence} \quad v = \frac{a}{A}V.$$

Inserting this value of v in the equation of energy, and solving for V, gives the result

$$V = \sqrt{\frac{2gh}{1 - \left(\frac{a}{A}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is always greater than the value $\sqrt{2gh}$.

The influence of a constantly maintained head on the velocity of flow at the orifice can now be ascertained by assigning values to the ratio $\frac{a}{A}$, thus:

$$\begin{array}{lll} \text{If} & a = A, & V = \infty; \\ \text{If} & a = \frac{2}{3}A, & V = 1.342\sqrt{2gh}; \\ \text{If} & a = \frac{1}{2}A, & V = 1.154\sqrt{2gh}; \\ \text{If} & a = \frac{1}{3}A, & V = 1.061\sqrt{2gh}; \\ \text{If} & a = \frac{1}{5}A, & V = 1.021\sqrt{2gh}; \\ \text{If} & a = \frac{1}{10}A, & V = 1.005\sqrt{2gh}. \end{array}$$

It is here indicated that the common formula (8) is in error 2.1 per cent when $a = \frac{1}{5}A$, if the head be maintained constant by a uniform vertical inflow at the water surface, and 0.5 per cent when $a = \frac{1}{10}A$. Practically, if the area of the orifice be less than one-twentieth of the cross-section of the vessel, the error in using the formula $V = \sqrt{2gh}$ is too small to be noticed even in the most precise experiments, and fortunately most orifices are smaller in relative size than this.

A more common case is that where the reservoir is of large

horizontal and small vertical cross-section, and where the water approaches the orifice with a horizontal velocity, as in a canal or trough. Here let A be the area of the vertical cross-section of the vessel, a the area of the orifice, and h the head upon its centre.

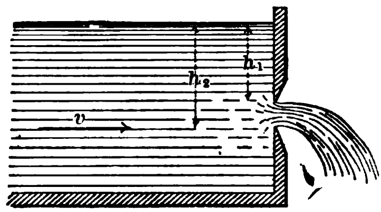

FIG. 16.

Then if h be large compared with the depth of the orifice, exactly the same reasoning applies as before, and the theoretic velocity of flow is

$$V = \sqrt{\frac{2gh}{1 - \left(\frac{a}{A}\right)^2}}.$$

If, however, h be small, let h_1 and h_2 be the heads on the upper and lower edges of the orifice, which is taken as rectangular. Then, using the same reasoning as above, the velocity of flow at any depth y is given by

$$V^2 = 2gy + v^2,$$

where v is the constant velocity of approach through the area A. The discharge through a strip of the length b and depth δy (Art. 20) then is

$$\delta Q = b\delta y(2gy + v^2)^{\frac{1}{2}},$$

and, by integration between the limits h_1 and h_2, the theoretic discharge per second from the orifice is

$$Q = \tfrac{2}{3} b \sqrt{2g}\left[\left(h_2 + \frac{v^2}{2g}\right)^{\frac{3}{2}} - \left(h_1 + \frac{v^2}{2g}\right)^{\frac{3}{2}}\right] \quad . \quad . \quad . \quad (11)'$$

In this case, particularly when $h_1 = 0$, the velocity of approach may exercise a marked influence on the discharge.

Prob. 33. In the case of horizontal approach, as seen in Fig. 16, let $b = 4$ feet, $h_2 = 0.8$ feet, $h_1 = 0$, and $v = 2.5$ feet per second. Compute the theoretic discharge: first, neglecting v; and second, regarding v.

ARTICLE 26. FLOW UNDER PRESSURE.

The level of water in the reservoir and the orifice of outflow have been thus far regarded as subjected to no pressure, or at least only to the pressure of the atmosphere which acts upon both with the same mean force of 14.7 pounds per square inch (since the head h is rarely or never so great that a sensible variation in atmospheric pressure can be detected between the orifice and the water level). But the upper level of the water may be subject to the pressure of steam or to the pressure due to a heavy weight or to a piston. The orifice may also be under a pressure greater or less than that of the atmosphere. It is required to determine the velocity of flow from the orifice under these conditions.

First, suppose that the surface of the water in the vessel or reservoir is subjected to the uniform pressure of p_0 pounds per square foot above the atmospheric pressure, while the pressure at the orifice is the same as that of the atmosphere. Let h be the depth of water on the orifice. The velocity of flow V is greater than $\sqrt{2gh}$ on account of the pressure p_0, and it is evidently the same as that from a column of water whose height is such as to produce the same pressure at the orifice. The total unit-pressure at the depth of the orifice is

$$p = wh + p_0,$$

and from (1) the head of water which would produce this pressure is

$$\frac{p}{w} = h + \frac{p_0}{w}.$$

Accordingly the velocity of flow from the orifice is

$$V = \sqrt{2g\left(h + \frac{p_0}{w}\right)},$$

or, if h_0 denote the head corresponding to the pressure p_0,

$$V = \sqrt{2g(h + h_0)}.$$

The general formula (6) thus applies to any small orifice, if h be the head corresponding to the static pressure at the orifice.

Secondly, suppose that the surface of the water in the vessel is subjected to the unit-pressure p_0, while the orifice is under the external unit-pressure p_1. Let h be the head of actual water on the orifice, h_0 the head of water which will produce the pressure p_0, and h_1 the head which will produce p_1. The velocity of flow at the orifice is then the same as if the orifice were under a head $h + h_0 - h_1$, or

$$V = \sqrt{2g(h + h_0 - h_1)}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which the values of h_0 and h_1 are

$$h_0 = \frac{p_0}{w}, \qquad h_1 = \frac{p_1}{w}.$$

Usually p_0 and p_1 are given in pounds per square inch, while h_0 and h_1 are required in feet; then (Art. 9)

$$h_0 = 2.304\,p_0, \qquad h_1 = 2.304\,p_1.$$

The values of p_0 and p_1 may be absolute pressures, or merely pressures above the atmosphere. In the latter case p_1 may sometimes be negative, as in the discharge of water into a condenser.

As an illustration of these principles let a cylindrical reser-

voir, Fig. 17, be 2 feet in diameter, and upon the surface of the water let there be a tightly fitting piston which with the load W weighs 3000 pounds. At the depth 8 feet below the water level are three small orifices: one at A, upon which there is an exterior head of water of 3 feet; one not shown in the figure, which discharges directly into the atmosphere; and one at C, where the discharge is into a vessel in which the tension of the air is only 10 pounds per square inch. It is required to determine the velocity of efflux from each orifice. The head h_0 corresponding to the pressure on the upper water surface is

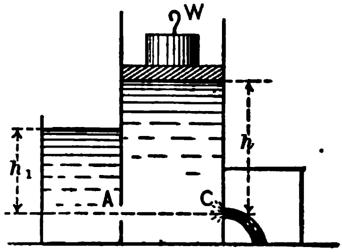

FIG. 17.

$$h_0 = \frac{p_0}{w} = \frac{3000}{3.1416 \times 62.5} = 15.28 \text{ feet.}$$

The head h_1 is 3 feet for the first orifice, 0 for the second, and $-2.304(14.7 - 10) = -10.83$ feet for the third. The three theoretic velocities of outflow then are:

$$V = 8.02\sqrt{8 + 15.28 - 3} = 36.1 \text{ feet per second;}$$
$$V = 8.02\sqrt{8 + 15.28 - 0} = 38.7 \text{ feet per second;}$$
$$V = 8.02\sqrt{8 + 15.28 + 10.83} = 46.8 \text{ feet per second.}$$

In the case of discharge from an orifice under water, as at A in Fig. 17, the value of $h - h_1$ is the same wherever the orifice be placed below the lower level, and hence the velocity depends upon the difference of level of the two water surfaces, and not upon the depth of the orifice.

The velocity of flow of oil or mercury under pressure is to be determined in the same manner as water, by finding the

heads which will produce the given pressure. Thus in the preceding numerical example, if the liquid be mercury, whose weight per cubic foot is 850 pounds, the head of mercury corresponding to the pressure of the piston is

$$h_0 = \frac{3000}{3.1416 \times 850} = 1.12 \text{ feet},$$

and, accordingly, for discharge into the atmosphere at the depth $h = 8$ feet the velocity is

$$V = 8.02 \sqrt{8 + 1.12} = 24.2 \text{ feet per second},$$

while for water the velocity was 38.7 feet per second. The general formula (6) is applicable to all cases of the flow of liquids from a small orifice, if for h its value $\frac{p}{w}$ be substituted, where p is the resultant unit-pressure at the depth of the orifice, and w is the weight of a cubic unit of the liquid.

Prob. 34. Water under a head of 230 feet flows into a boiler whose gauge reads 45 pounds per square inch. Find the velocity of the inflowing water.

Prob. 35. The pressure in a boiler is 60 pounds per square inch above the atmosphere. Compute the theoretic velocity of flow from a small orifice one foot below the water level.

ARTICLE 27. PRESSURE-HEAD AND VELOCITY-HEAD.

When a vessel is filled with water at rest the pressure at any point depends only upon the head of water above that point (Art. 9). But when the water is in motion it is a fact of observation that the pressure becomes less than that due to the head. The actual pressure in any event may be measured by the height of a column of water. Thus if the water be at

rest in the case shown in Fig. 18, and small tubes be inserted at A, B, and C, the water will rise in each tube to the same height as that of the water level in the reservoir, and the pressures at A, B, and C will be those due to the heads Aa, Bb, and Cc. But if an orifice be opened, as seen near C, the water levels in the tubes sink to the points a_1, b_1, and c_1; that is, the pressures at A, B, and C are reduced to those due to the heads Aa_1, Bb_1, and Cc_1.

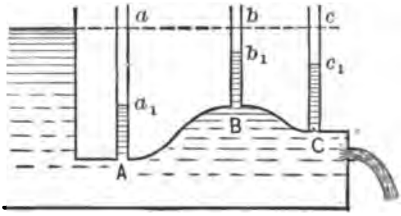

FIG. 18.

Let h be the head of water on any point, or the depth of that point below the free water level. Let h_1 be the head due to the actual pressure of the water at that point, or the pressure-head. Let $\frac{v^2}{2g}$ be the head due to the actual velocity of the water at that point, or the velocity-head. Then

$$h_1 + \frac{v^2}{2g} = h; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

or, in the form of a theorem:

> The pressure-head plus the velocity-head is equal to the total hydrostatic head.

In order to prove this let W be the weight of water which passes the section per second; then $W\frac{v^2}{2g}$ is the energy which it possesses. The total theoretic energy of this water is Wh, and if there be no losses of energy the remaining energy is $W\left(h - \frac{v^2}{2g}\right)$, which is to be equated to Wh_1, which represents the potential energy still existing in the form of pressure.

Hence
$$h - \frac{v^2}{2g} = h_1,$$

whence the theorem follows as stated. In Fig. 18 aa_1 is the velocity-head for the section A, while Aa_1 is the pressure-head.

Another method of proof is to consider the section at A as an orifice through which the flow occurs under a head $h - h_1$, where h_1 is the head caused by the back pressure p_1. Then, from the last article,

$$v = \sqrt{2g(h - h_1)},$$

from which $\frac{v^2}{2g} = h - h_1$, which also agrees with the theorem.

The pressure-head Aa_1 at A hence decreases when the velocity of the water at A increases, and the same is true for any other section as B. Let v and v' be the velocities at A and B; then, since the same quantity of water passes each section per second, the relation $Av = Bv'$ must be fulfilled. Hence if B be greater than A the velocity v is greater than v', and the pressure-head at B will be greater than at A. To illustrate: let the depths of A and B be 6 and 5 feet respectively below the water level, and the corresponding cross-sections be 1.2 and 2.4 square feet. Let the quantity of water discharged by the orifice near C be 14.4 cubic feet per second. Then the velocity at A is

$$v = \frac{14.4}{1.2} = 12 \text{ feet per second,}$$

which corresponds to a velocity-head of

$$\frac{v^2}{2g} = 0.01555v^2 = 2.24 \text{ feet};$$

and accordingly the pressure-head Aa_1 is

$$h_1 = 6.0 - 2.24 = 3.76 \text{ feet.}$$

Proceeding in the same way for B, the velocity is found to be 6 feet per second, the velocity-head 0.56 feet, and finally the pressure-head is $5.0 - 0.56 = 4.44$ feet. The hydrostatic head at A is thus diminished by the velocity-head $aa_1 = 2.24$ feet, while at B it is diminished by the smaller amount $bb_1 = 0.56$ feet. When the water was at rest the pressures were:

At A, $p = 0.434 \times 6 = 2.60$ pounds per square inch;
At B, $p = 0.434 \times 5 = 2.17$ pounds per square inch.

But as soon as the flow from the orifice began the pressures became:

At A, $p = 0.434 \times 3.76 = 1.63$ pounds per square inch;
At B, $p = 0.434 \times 4.44 = 1.93$ pounds per square inch.

A negative pressure may occur if the velocity-head becomes greater than the hydrostatic head; for since $h_1 + \frac{v^2}{2g}$ equals h, the value of h_1 is negative when $\frac{v^2}{2g}$ exceeds h. A case in which this may occur is shown in Fig. 19, where the section at A is so small that $\frac{v^2}{2g}$ becomes larger than h, so that if a tube be inserted no water runs out, but if the tube be carried downward into a vessel of water there will be lifted a column CD whose height is that of the negative pressure-head h_1. For example, let the cross-section of A be 0.4 square feet, and its head h be 4.1 feet, while 8 cubic feet per second are discharged from the orifice below. Then the velocity at A is 20 feet per second, and the corresponding velocity-head is

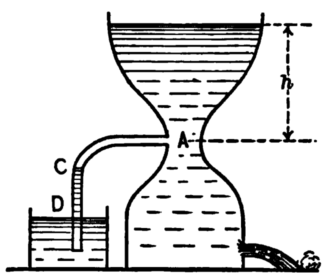

FIG. 19.

6.22 feet. The pressure-head at A then is, from (13),

$$h_1 = 4.1 - 6.22 = -2.12 \text{ feet,}$$

and accordingly there exists at A an inward, or negative, pressure,

$$p_1 = -2.12 \times 0.434 = -0.92 \text{ pounds per square inch.}$$

This negative pressure will sustain a column of water CD whose height is 2.12 feet. If the small vessel be placed so that its water level is less than 2.12 feet below, water will be constantly drawn from the smaller to the larger vessel. This is the principle of the action of the injector-pump.

Prob. 36. The hydrostatic pressure in a pipe is 80 pounds per square inch. What velocity must the water have to reduce this to 50 pounds per square inch?

ARTICLE 28. TIME OF EMPTYING A VESSEL.

Let the depth of water in a vessel be H; it is required to determine the time of emptying it through a small orifice in the base whose area is a. Let Y be the area of the water surface when the depth of water is y; let δt be the time during which the water level falls the distance δy. During this time the quantity of water $Y\delta y$ passes through the orifice. But the discharge in one second under the constant head y is $a\sqrt{2gy}$, and hence the discharge in the time δt is $a\delta t\sqrt{2gy}$. Equating these two expressions, there is found the relation

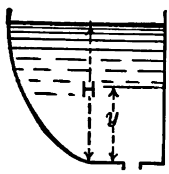

FIG. 20.

$$\delta t = \frac{Y\delta y}{a\sqrt{2gy}}.$$

The time of emptying the vessel is now found by inserting for Y its value in terms of y, and then integrating between the limits H and 0.

For a cylinder or prism the cross-section Y has the constant value A, and the formula becomes

$$\delta t = \frac{Ay^{-\frac{1}{2}}\delta y}{a\sqrt{2g}},$$

the integration of which gives

$$t = \frac{2A\sqrt{H}}{a\sqrt{2g}} = \frac{2AH}{a\sqrt{2gH}}$$

as the theoretic time of emptying the vessel. If the head were maintained constant the uniform discharge per second would be $a\sqrt{2gH}$, and the time of discharging a quantity equal to the capacity of the vessel is AH divided by $a\sqrt{2gH}$, which is one half of the time required to empty it.

To find the time of emptying a hemispherical bowl of radius r, let x be the radius of the cross-section Y; then

$$x^2 + (r-y)^2 = r^2;$$
$$x^2 = 2ry - y^2;$$
$$Y = \pi(2ry - y^2).$$

The equation for δt then becomes

$$\delta t = \frac{\pi}{a\sqrt{2g}}(2ry^{\frac{1}{2}} - y^{\frac{3}{2}})\delta y,$$

and by integration between the limits r and 0

$$t = \frac{14\pi r^{\frac{5}{2}}}{15a\sqrt{2g}},$$

which is the theoretic time required to empty the hemisphere.

The only important application of these principles is in the case of the right prism or cylinder, and the formula for this is materially modified in practice, as will be seen in the next chapter. It is more frequently required to determine the time during which the water level will descend from the

height H to another height h. This is found by integrating between the limits H and h; thus, for the prismatic vessel,

$$t = \frac{2A}{a\sqrt{2g}}(\sqrt{H} - \sqrt{h}), \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which gives the theoretic time of descent in seconds.

Prob. 37. A sphere is filled with water. Find the time of emptying it through a small orifice at its lowest point.

Prob. 38. A conical vessel whose altitude is H, and whose base has the radius r, is placed with its axis vertical, and emptied through a small orifice in its base. Prove that the theoretic time is $\frac{16\pi r^2\sqrt{H}}{15a\sqrt{2g}}$.

ARTICLE 29. FLOW FROM A REVOLVING VESSEL.

The water in a vessel at rest is acted upon only by the force of gravity, and hence its surface is a horizontal plane; but the water in a revolving vessel is acted upon by a centrifugal force as well as by gravity, so that its surface assumes a curved shape. The simplest case is that of a vessel revolving with uniform velocity about a vertical axis, and it will be shown that here the water surface forms a paraboloid whose axis coincides with that about which it revolves. Fig. 21 represents such a case, NT being the vertical axis.

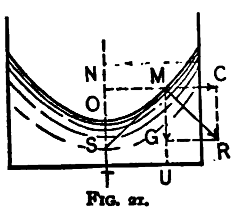

FIG. 21.

Let M be any point on the surface whose co-ordinates ON and NM are y and x. Let W be the weight of a particle at M, whose intensity is represented by MG; this particle in consequence of its velocity of revolution u is acted upon also by a centrifugal force MC whose value* is $\frac{W}{g} \cdot \frac{u^2}{x}$. The resultant

* See WOOD'S Elementary Mechanics, p. 226.

MR of the weight and centrifugal force must be normal to the tangent MS at M, as the condition of equilibrium. The angle NMS is hence equal to RMG, and accordingly

$$\tan NMS = \frac{MC}{MG} = \frac{u^2}{gx}.$$

But the tangent of this angle is the first derivative of y with reference to x. Further, the value of u varies directly with x, so that $u = \omega x$ if ω be the angular velocity, that is, the velocity at the distance unity from the axis. Accordingly,

$$\frac{\delta y}{\delta x} = \frac{u^2}{gx} = \frac{\omega^2}{g} x$$

is the differential equation of the curve, and by integration

$$y = \frac{\omega^2 x^2}{2g},$$

which is the equation of a common parabola. Therefore the surface is a paraboloid. Since ωx is the velocity u at the point M, this equation may be written

$$y = \frac{u^2}{2g},$$

which shows that the ordinate y is the head due to the velocity of revolution.

If h be the head OT at the axis, the velocity of efflux from a small orifice at T is $\sqrt{2gh}$. But for an orifice at U the velocity is due to the head MU, and

$$MU = OT + NO = h + y.$$

The theoretic velocity of flow from U therefore is

$$V = \sqrt{2g(h + y)} = \sqrt{2gh + u^2}, \quad . \quad . \quad . \quad . \quad (15)$$

where u is the velocity of revolution of the point U or M. This formula is a very important one in the discussion of certain hydraulic motors.

To determine the velocity u of a point at the distance x from the axis of revolution it is only necessary to count the number of revolutions made per second. If n be this number,

$$u = 2\pi x \, . \, n;$$

or, in another form, since $2\pi n$ is the velocity at the distance unity from the axis,

$$\omega = 2\pi n \quad \text{and} \quad u = \omega x.$$

As an example of the application of these principles, let there be a cylindrical vessel which is 2 feet in diameter and 3 feet deep, and which is one half full of water. It is required to find the number of revolutions per second about its axis which will cause the water to begin to overflow around the upper edge. The volume of a paraboloid being one-half of its circumscribing cylinder, the vertex of the paraboloid at the moment of overflow will coincide with the centre of the base of the vessel, and hence the value of y for the upper edge is 3 feet. Accordingly,

$$y = 3 = \frac{\omega^2 . 1^2}{2g},$$

whence $\omega = 13.89$, and then

$$n = \frac{13.89}{2\pi} = 2.21,$$

which is the number of revolutions per second. If the vessel were three-fourths full of water, the volume of the paraboloid at the moment of overflow would be one-fourth that of the cylinder, and the value of y for the upper edge would be one-half the altitude of the cylinder, or 1.5 feet. Hence ω is found to be 9.82, whence the number of revolutions per second is about 1.56.

Prob. 39. A cylindrical vessel is 3 feet in diameter. How many revolutions per minute must be made about its vertical

axis in order that the velocity of the outer surface may be 50 feet per second?

Prob. 40. A cylindrical vessel 2 feet in diameter and 3 feet deep is three-fourths full of water, and is revolved about its vertical axis so that the water is just on the point of overflowing around the upper edge. Find the theoretic velocity of efflux from an orifice in the base at a distance of 9 inches from the axis. Ans. 12.28 feet per second.

ARTICLE 30. THE PATH OF A JET.

When a jet of water issues from a small orifice in the vertical side of a vessel or reservoir, its direction at first is horizontal, but the force of gravity immediately causes the jet to move in a curve which will be shown to be the common parabola. Let x be the abscissa and y the ordinate of any point of the curve, measured from the orifice as an origin, as seen in Fig 22. The effect of the impulse at the orifice is to cause the space x to be described uniformly in a certain time t, or, if v be the velocity of flow, $x = vt$. The effect of the force of gravity is to cause the space y to be described in accordance with the laws of falling bodies (Art. 6), or $y = \frac{1}{2}gt^2$. Eliminating t from these two equations gives

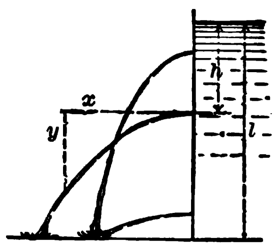

FIG. 22.

$$y = \frac{gx^2}{2v^2} = \frac{x^2}{4h},$$

which is the equation of a parabola whose axis is vertical and whose vertex is at the orifice.

The horizontal range of the jet for any given ordinate y is found from the equation $x^2 = 4hy$. If the height of the vessel be l, the horizontal range on the plane of the base is

$$x = 2\sqrt{h(l - h)}.$$

This value is o when $h = 0$ and also when $h = l$, and it is a maximum when $h = \frac{1}{2}l$. Hence the greatest range is from an orifice at the mid-height of the vessel.

A more general case is that where the side of the vessel is inclined to the vertical at the angle θ, as in Fig. 23. Here the jet at first issues perpendicularly to the side, and under the action of the impulsive force a particle of water would describe the distance AB in a certain time t. But in that same time the force of gravity causes it to descend through the distance BC. Now let x be the horizontal abscissa and y the vertical ordinate of the point C measured from the origin A. Then $AB = x \sec \theta$, and $BC = x \tan \theta - y$. Hence

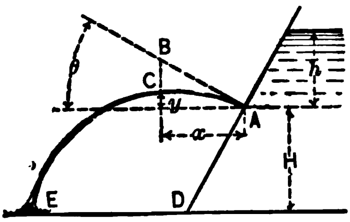

FIG. 23.

$$x \sec \theta = vt, \qquad x \tan \theta - y = \tfrac{1}{2}gt^2.$$

The elimination of t from these expressions gives, after replacing v^2 by its value $2gh$,

$$y = x \tan \theta - \frac{x^2 \sec^2 \theta}{4h}, \quad \ldots \ldots \quad (16)$$

which is also the equation of a common parabola.

To find the horizontal range in the level of the orifice make $y = 0$; then

$$x = 4h \frac{\tan \theta}{\sec^2 \theta} = 2h \sin 2\theta.$$

This is o when $\theta = 0°$ or $\theta = 90°$; it is a maximum and equal to $2h$ when $\theta = 45°$. To find the highest point of the jet the first derivative of y with reference to x is to be equated to zero

in order to locate the point where the tangent to the curve is horizontal; thus,

$$\frac{\delta y}{\delta x} = \tan\theta - \frac{x \sec^2\theta}{2h} = 0,$$

from which $x = 2h \sin\theta \cos\theta$, and this, inserted in the equation of the curve, gives

$$y = h \sin^2\theta,$$

which is the highest elevation of the jet above the orifice. In this, if $\theta = 90°$, $y = h$; that is, if a jet be directed vertically upward it will, theoretically, rise to the height of the level of water in the reservoir.

As a numerical example let a vessel whose height is 16 feet stand upon a horizontal plane *DE*, Fig. 23, the side of the vessel being inclined to the vertical at the angle $\theta = 30°$. Let a jet issue from a small orifice at *A*, under a head of 10 feet. The jet rises to its maximum height, $y = \frac{1}{4}10 = 2.5$ feet, at the distance $x = \frac{1}{2}\sqrt{3} \times 10 = 8.66$ feet from *A*. At $x =$ 17.32 feet the jet crosses the horizontal plane through the orifice. To find the point where it strikes the plane *DE*, the value of y is made -6 feet; then, from the equation of the curve,

$$-6 = x\sqrt{\tfrac{1}{3}} - \frac{x^2}{30},$$

from which x is found to be 24.62 feet; whence, finally,

$$DE = 24.62 - 6 \tan 30° = 21.16 \text{ feet.}$$

Prob. 41. Find all the circumstances of the motion of a jet which issues from a vessel under a head of 5 feet, the side of the vessel being inclined to the vertical at an angle of 60°, and its depth being 9 feet.

ARTICLE 31. THE ENERGY OF A JET.

Let a jet or stream of water have the velocity v, and let W be the weight of water per second passing any given cross-section. The energy of this moving water, or the work which it is capable of doing, is the same as that stored up by a body falling freely under the action of gravity through a height h and thereby acquiring the velocity v. Thus, if K be the energy or potential work,

$$K = Wh = W\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Therefore, for a constant quantity of water per second passing through the given cross-section, the energy of the jet is proportional to the square of its velocity.

The weight W, however, may be expressed in terms of the cross-section of the jet and its velocity. Thus, if a be the area of the cross-section, and w the weight of a cubic unit of water, W is the weight of a column of water whose length is v and whose cross-section is a, or $W = wav$; and hence (17) may be written

$$K = \frac{wav^3}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)'$$

In general, then, it may be stated that for a constant cross-section, the energy of a jet, or the work which it is capable of doing per second, varies with the cube of its velocity.

The expressions just deduced give the theoretic energy of the jet, that is, the maximum work which can be obtained from it; but this in practice can never be fully utilized. The amount of work which is realized when a jet strikes a moving surface,

like the vane of a water-motor, depends upon a number of circumstances which will be explained in a later chapter, and it is the constant aim of inventors so to arrange the conditions that the actual work may be as near to the theoretic energy as possible. The "efficiency" of an apparatus for utilizing the energy of moving water is the ratio of the work actually utilized to the theoretic work; or, if k be the work realized, the efficiency e is

$$e = \frac{k}{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The greatest possible value of e is unity, but this can never be attained, owing to the imperfections of the apparatus and the hurtful resistances. Values greater than 0.90 have, however, been obtained; that is, 90 per cent or more of the theoretic work has been utilized in some of the best forms of hydraulic motors.

For example, let water issue from a pipe 2 inches in diameter with a velocity of 10 feet per second. The cross-section in square feet is $\frac{3.1416}{144}$, and the theoretic work in foot-pounds per second is

$$K = 0.01555 \times 62.5 \times 0.0218 \times 10^3 = 21.2,$$

which is 0.0385 horse-powers. If the velocity is 100 feet per second, however, the theoretic horse-power of the stream will be 38.5.

Prob. 42. One cubic foot of water per second flows from an orifice with a velocity of 32 feet per second. Find the theoretic horse-power of the stream.

Prob. 43. A small turbine wheel using 102 cubic feet of water per minute under a head of 40 feet is found to give 6.15 horse-power. Find the efficiency of the wheel.

Ans. 80 per cent.

ARTICLE 32. THE IMPULSE AND REACTION OF A JET.

When a stream or jet is in motion delivering W pounds of water per second with the uniform velocity v, that motion may be regarded as produced by a constant impulsive force F, which has acted upon W for one second and then ceased. In this second the velocity of F has increased from 0 to v, and the space $\frac{1}{2}v$ has been described. Consequently the work $F \times \frac{1}{2}v$ has been imparted to the water by the impulse F. But the theoretic energy of the jet is $W\frac{v^2}{2g}$; hence

$$F \times \tfrac{1}{2}v = W\frac{v^2}{2g},$$

from which the force of impulse F is

$$F = W\frac{v}{g}. \quad \dots \dots \dots \quad (19)$$

Let a be the area of the cross-section of the jet; then $W = wav$, and

$$F = wa\frac{v^2}{g}. \quad \dots \dots \dots \quad (19)'$$

Therefore the impulse of a jet of constant cross-section varies as the square of its velocity.

The force F is a continuous impulsive pressure acting in the direction of the motion. For, by the definition, F acts for one second upon the W pounds of water which pass a given section; but in the next second W pounds also pass the section, and the same is the case for each second following. This impulse will be exerted as a pressure upon any surface placed in the path of the jet.

The reaction of a jet upon a vessel occurs when water flows from an orifice. This reaction must be equal in value and opposite in direction to the impulse, as in all cases of stress

action and reaction are equal. In the direction of the jet the impulse produces motion, in the opposite direction it produces a pressure which tends to move the vessel. The force of reaction of a jet hence is

$$F = W\frac{v}{g} = wa\frac{v^2}{g}.$$

To compare this with hydrostatic pressure, let h be the velocity-head due to v; then

$$F = 2wa\frac{v^2}{2g} = 2wah.$$

But, from Art. 10, the normal pressure on a surface of area a under the hydrostatic head h is wah. Therefore the dynamic pressure caused by the reaction of a jet issuing from an orifice in a vessel is double the hydrostatic pressure on the orifice when closed. This theoretic conclusion has been verified by experiment.

The full force of impulse or reaction is exerted in the line of the action of the jet, and its force in any other direction is the component of the force F in that direction. Hence in a direction which makes an angle θ with the line of motion of the jet, the force which can be exerted by the impulse or reaction is $F\cos\theta$. Thus if water issues from an orifice in the base of a vessel, it exerts an upward reaction F and a horizontal reaction 0; if it issues in a direction inclined 30° to the vertical, its upward reaction is $F\cos 30°$, and its horizontal reaction is $F\sin 30°$.

If a stream moving with the velocity v_1 is retarded so that its velocity becomes v_2, its impulse in the first instance is $W\frac{v_1}{g}$, and in the second $W\frac{v_2}{g}$. The difference of these, or

$$P = W\frac{v_1 - v_2}{g}$$

is a measure of the dynamic pressure developed. It is by virtue of the pressure due to change of velocity that turbine wheels and other hydraulic motors transform the energy of moving water into useful work.

Prob. 44. Devise an experiment for measuring the force of reaction of a jet which issues from an orifice in the base or side of a vessel.

ARTICLE 33. ABSOLUTE AND RELATIVE VELOCITIES.

Absolute velocity is that with respect to the earth, and relative velocity that with respect to a body in motion. For instance, if water issues from a small orifice in a vessel which is in motion in a straight line with the uniform velocity u, the theoretic velocity of flow relative to the vessel is $V = \sqrt{2gh}$, or the same as its absolute velocity if the vessel were at rest, for no accelerating forces exist to change the direction or the value of g. The absolute velocity of flow, however, may be greater or less than V, depending upon the value of u and its direction. To illustrate: Fig. 24 shows a moving vessel from which water is flowing through three orifices. At A the direction of V is horizontal, and as the vessel is moving in the opposite direction with the velocity u, the absolute velocity of the water as it leaves the orifice is

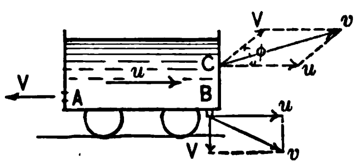

FIG. 24.

$$v = V - u.$$

It is plain that if the orifice were in the front of the vessel and the direction of V were horizontal, the absolute velocity would be $v = V + u$.

Again, at B is an orifice from which the water issues vertically with respect to the vessel with the relative velocity V,

while at the same time the orifice moves horizontally with the velocity u. Forming the parallelogram, the absolute velocity v is seen to be the resultant of V and u, or

$$v = \sqrt{V^2 + u^2}.$$

Lastly, at C is shown an orifice in the front of the vessel so arranged that the direction of the relative velocity V makes an angle ϕ with the horizontal. From C draw Cu to represent the velocity u, and CV to represent V, and complete the parallelogram as shown; then Cv, the resultant of u and V, is the absolute velocity with which the water leaves the orifice. From the triangle Cuv,

$$v = \sqrt{u^2 + V^2 + 2uV\cos\phi}.$$

In this, if $\phi = 0$, v becomes $u + V$ as before shown; if $\phi = 90°$, it becomes the same as when the water issues vertically from the orifice in the base; and if $\phi = 180°$, the value of v is that before found for an orifice in the side of the vessel.

In Art. 29 the velocity of flow from an orifice in a vessel revolving with uniform velocity was found to be

$$V = \sqrt{2gh + u^2}.$$

This is the velocity relative to the vessel. If the orifice be in the base, the direction of V with respect to the vessel is vertical, and as the orifice is moving horizontally with the uniform velocity u, the absolute velocity of flow is

$$v = \sqrt{u^2 + V^2} = \sqrt{2gh + 2u^2}.$$

In the same way, if the orifice be in the side of the vessel, and the direction of V be horizontal and directly away from the axis, the same formula applies, for the absolute velocity v is the resultant of the two rectangular components V and u.

If a vessel move with a motion which is accelerated or retarded, this affects the value of g, and the reasoning of the preceding articles does not give the correct value of V. For instance, if a vessel move vertically upward with an acceleration f, the theoretic relative velocity of flow from an orifice in it is

$$V = \sqrt{2(g + f)h};$$

and if u be its velocity at any instant, the absolute velocity of flow is $u + V$. This equation shows that if a vessel be moving downward with the acceleration g, that is, freely falling, V will be zero, which of course is to be expected since both water and vessel are alike accelerated.

Prob. 45. If V be velocity of flow from the orifice at A in Fig. 23, show that the velocity of the jet at the point E is $\sqrt{V^2 + 2gH}$.

Prob. 46. If a vessel of water is moving horizontally with an acceleration $\frac{1}{4}g$, show that the surface of the water is a plane which is inclined to the horizontal at an angle of about 14 degrees.

CHAPTER IV.

FLOW OF WATER THROUGH ORIFICES.

ARTICLE 34. THE STANDARD ORIFICE.

Orifices for the measurement of water are usually placed in the vertical side of a vessel or reservoir, but may also be placed in the base. In the former case it is understood that the upper edge of the opening is completely covered with water; and generally the head of water on an orifice is at least three or four times its vertical height. The term standard orifice is here used to signify that the opening is so arranged that the water in flowing from it touches only a line, as would be the case in a plate of no thickness. To secure this result the inner edge of the opening has a square corner, which alone is touched by the water. In precise experiments the orifice may be in a metallic plate whose thickness is really small, as at *A* in Fig. 25, but more commonly it is cut in a board or plank, care being taken that the inner edge is a definite corner. It is usual to bevel the outer edges of the orifice as at *C*, so that the escaping jet may by no possibility touch the edges except at the inner corner. The term "orifice in a thin plate" is often used to express the condition that the water shall only touch the edges of the opening along a line. This arrange-

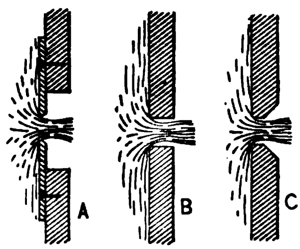

FIG. 25.

ment may be regarded as a kind of standard apparatus for the measurement of water, for, as will be seen later, the discharge is modified if the inner corner is rounded, and different degrees of rounding give different discharges. Orifices arranged as in Fig. 25 are accordingly always used when water is to be measured by the use of orifices.

The contraction of the jet which is always observed when water issues from a standard orifice as described above is a most interesting and important phenomenon. It is due to the circumstance that the particles of water as they approach the orifice move in converging directions, and that these directions continue to converge for a short distance beyond the plane of the orifice. It is this contraction of the jet that causes only the inner corner of the orifice to be touched by the escaping water. The appearance of such a jet under steady flow, issuing from a circular orifice, is that of a clear crystal bar whose beauty excites the admiration of every observer. The place of greatest contraction is at a distance from the plane of the orifice of about one-half its diameter, and beyond this point the jet gradually enlarges in size, while its surface becomes more or less disturbed owing to the resistance of the air and other causes. In the case of square and rectangular orifices the contraction of the jet is also observed, its edges being angular and its cross-section similar to that of the orifice until the place of greatest contraction is passed.

Owing to this contraction the discharge from a standard orifice is always less than the theoretic discharge. It is the object of this chapter to determine how the theoretic formulas are to be modified so that they may be used for the practical purposes of the measurement of water. This is to be done by the discussion of the results of experiments. It will be supposed, unless otherwise stated, that the size of the orifice is small compared with the cross-section of the reservoir, so that

the effect of velocity of approach may be neglected (Art. 25).

Prob. 47. Under a head of 6 feet the discharge from an orifice is 3.74 gallons per second. What head is necessary in order that the discharge may be one cubic foot per second?

ARTICLE 35. THE COEFFICIENT OF CONTRACTION.

The coefficient of contraction is the number by which the area of the orifice is to be multiplied in order to give the area of the least cross-section of the jet. Thus, if c' be the coefficient of contraction, a the area of the orifice, and a' that of the jet,

$$a' = c'a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The coefficient of contraction is evidently always less than unity.

The only direct method of finding the value of c' is to measure by callipers the dimensions of the least cross-section of the jet. The size of the orifice can usually be determined with accuracy, but no great precision can be attained in measuring the jet. To find c' for a circular orifice let d and d' be the diameters of the sections a and a'; then

$$c' = \frac{a'}{a} = \left(\frac{d'}{d}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)'$$

Therefore the coefficient of contraction is the square of the ratio of the diameter of the jet to that of the orifice. In this way NEWTON found for c' the value 0.71; BORDA, 0.65; BOSSUT, from 0.66 to 0.67; MICHELOTTI, from 0.57 to 0.624 with a mean of 0.61. EYTELWEIN gave 0.64 as a mean value, and WEISBACH mentions 0.63.

As a mean value the following may be kept in mind by the student:

Coefficient of contraction $c' = 0.62$;

or, in other words, the minimum cross-section of the jet is 62 per cent of that of the orifice. This value, however, undoubtedly varies for different forms of orifices and for the same orifice under different heads, but little is known regarding the extent of these variations or the laws that govern them. Probably c' is slightly smaller for circles than for squares, and smaller for squares than for rectangles, particularly if the rectangle be long compared with its width. Probably also c' is larger for low heads than for high heads.

Prob. 48. The diameter of a circular orifice is 1.995 inches. Three measurements of the diameter of the least cross-section of the jet give the values 1.55, 1.56, and 1.59 inches. Find the coefficient of contraction.

ARTICLE 36. THE COEFFICIENT OF VELOCITY.

The coefficient of velocity is the number by which the theoretic velocity of flow from the orifice is to be multiplied in order to give the actual velocity at the least cross-section of the jet. Thus, if c_1 be the coefficient of velocity, V the theoretic velocity due to the head on the centre of the orifice, and v the actual velocity at the contracted section,

$$v = c_1 V = c_1 \sqrt{2gh}. \qquad (21)$$

The coefficient of velocity must be less than unity, since the force of gravity cannot generate a greater velocity than that due to the head.

The velocity of flow at the contracted section of the jet cannot be directly measured. To obtain the value of the coefficient of velocity, indirect observations have been taken on the path of the jet. Referring to Art. 30, it will be seen that when a jet flows from an orifice in the vertical side of a vessel it takes a path whose equation is

$$y = \frac{gx^2}{2v^2},$$

in which x and y are the co-ordinates of any point of the path measured from vertical and horizontal axes, and v is the velocity at the origin. Now placing for v its value $c_1\sqrt{2gh}$, and solving for c_1, gives

$$c_1 = \frac{x}{2\sqrt{hy}}.$$

Therefore c_1 becomes known by the measurement of the two co-ordinates x and y and the head h.

In conducting this experiment it would be well to have a ring, a little larger than the jet, supported by a stiff frame which can be moved until the jet passes through the ring. The flow of water can then be stopped, and the co-ordinates of the centre of the ring determined. By placing the ring at different points of the path different sets of co-ordinates can be obtained. The value of x should be measured from the contracted section rather than from the orifice, since v is the velocity at the former point and not at the latter.

By this method of the jet BOSSUT in two experiments found for the coefficient of velocity the values 0.974 and 0.980, MICHELOTTI in three experiments obtained 0.993, 0.998, and 0.983, and WEISBACH deduced 0.978. Great precision cannot be obtained in these determinations, nor indeed is it necessary for the purposes of hydraulic investigation that c_1 should be accurately known for standard orifices. As a mean value the following may be kept in the memory:

$$\text{Coefficient of velocity } c_1 = 0.98\,;$$

or, the actual velocity of flow at the contracted section is 98 per cent of the theoretic velocity. The value of c_1 is greater for high than for low heads, and may probably often exceed 0.99.

Another method of finding the coefficient c_1 is to place the orifice horizontal so that the jet will be directed vertically up-

ward as in Fig. 12, Art. 20. The height to which it rises is the velocity height h_0, or

$$h_0 = \frac{v^2}{2g},$$

in which v is the actual velocity $c_1\sqrt{2gh}$. Substituting, this value of v gives

$$h_0 = c_1^2 h,$$

from which, when h_0 is measured, c_1 is computed. For example, under a head of 23 feet a stream was found to rise to a height of 22 feet; then

$$c_1 = \sqrt{\frac{h_0}{h}} = \sqrt{\frac{22}{23}} = 0.978.$$

This method, like the preceding, fails to give good results for high velocities owing to the resistance of the air, and moreover it is impossible to measure with precision the height h_0.

Prob. 49. MICHELOTTI found the range of a jet to be 6.25 meters on a horizontal plane 1.41 meters below the vertical orifice, which was under a head of 7.19 meters. Compute the coefficient of velocity.

ARTICLE 37. THE COEFFICIENT OF DISCHARGE.

The coefficient of discharge is the number by which the theoretic discharge is to be multiplied in order to obtain the actual discharge. Thus, if c be the coefficient of discharge, Q the theoretical and q the actual discharge per second,

$$q = cQ. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Evidently c is a number less than unity.

The coefficient of discharge can be accurately found by allowing the flow from an orifice to fall into a vessel whose cubic contents are known with precision. The quantity q is

thus determined, while Q is computed from the formulas of the last chapter. Then

$$c = \frac{q}{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)'$$

For example, a circular orifice of 0.1 feet diameter was kept under a constant head of 4.677 feet; during a time of 5 minutes $32\frac{1}{5}$ seconds the jet flowed into a measuring vessel which was found to contain 27.28 cubic feet. Here the actual discharge per second was

$$q = \frac{27.28}{332.2} = 0.08212 \text{ cubic feet.}$$

The theoretic discharge, from formula (8), is

$$Q = \pi \times 0.05^2 \times 8.02 \sqrt{4.677} = 0.1361 \text{ cubic feet.}$$

Then, for the coefficient of discharge,

$$c = \frac{0.08212}{0.1361} = 0.604.$$

In this manner thousands of experiments have been made upon different forms of orifices under different heads, for accurate knowledge regarding this coefficient is of great importance in practical hydraulic work.

The following articles contain values of the coefficient of discharge for different kinds of orifices, and it will be seen that in general c is greater for low heads than for high heads, greater for rectangles than for squares, and greater for squares than for circles. Its value ranges from 0.59 to 0.63 or higher, and as a mean to be kept in mind by the student there may be stated:

$$\text{Coefficient of discharge } c = 0.61;$$

or, the actual discharge from orifices such as are shown in Fig. 25 is 61 per cent of the theoretic discharge.

The coefficient c may be expressed in terms of the coefficients c' and c_1. Let a and a' be the areas of the orifice and the cross-section of the contracted jet, and Q and q the theoretic and actual discharge per second. Then

$$c = \frac{q}{Q} = \frac{a' c_1 \sqrt{2gh}}{a \sqrt{2gh}} = \frac{a'}{a} c_1.$$

But (Art. 34) the ratio $a' : a$ is the coefficient c'; therefore

$$c = c'c_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or, the coefficient of discharge is the product of the coefficients of contraction and velocity.

Prob. 50. What is the discharge in gallons per minute from a circular orifice one inch in diameter under a head of 12 inches, the coefficient of discharge being 0.609?

Prob. 51. The diameter of a contracted circular jet was found to be 0.79 inches, the diameter of the orifice being one inch. Under a head of 4 feet the actual discharge per minute was found to be 3.21 cubic feet. Find the coefficient of velocity.

ARTICLE 38. CIRCULAR VERTICAL ORIFICES.

Let h be the head on the centre of a vertical circular orifice whose diameter is d. The theoretic discharge per second is found from formula (10), Art. 24, by placing for r its value $\frac{1}{2}d$, and the actual discharge per second is

$$q = c \, . \, \tfrac{1}{4}\pi d^2 \sqrt{2gh}\left(1 - \frac{1}{128}\frac{d^2}{h^2} - \frac{5}{16\,384}\frac{d^4}{h^4} - \frac{105}{4\,194\,304}\frac{d^6}{h^6} - \text{etc.}\right), \quad . \quad (24)$$

in which c is the coefficient of discharge. In case h becomes large compared with d, the negative terms in the parenthesis may be neglected, and

$$q = c.\tfrac{1}{4}\pi d^2 \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24')$$

which is the same as the formula for horizontal circular orifices (Art. 21).

The following table of values of c is abridged from the results deduced by HAMILTON SMITH, Jr.,* as determined by the discussion of all the best experiments. The table applies only to standard orifices.

TABLE VI. COEFFICIENTS FOR CIRCULAR VERTICAL ORIFICES.

Head h in Feet.	Diameter of Orifice in Feet.						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.637	0.624	0.618			
0.6	0.655	.630	.618	.613	0.601	0.593	
0.8	.648	.626	.615	.610	.601	.594	0.590
1.0	.644	.623	.612	.608	.600	.595	.591
1.5	.637	.618	.608	.605	.600	.596	.593
2.	.632	.614	.607	.604	.599	.597	.595
2.5	.629	.612	.605	.603	.599	.598	.596
3.	.627	.611	.604	.603	.599	.598	.597
4.	.623	.609	.603	.602	.599	597	.596
6.	.618	.607	.602	.600	.598	.597	.596
8.	.614	.605	.601	.600	.598	.596	.596
10.	.611	.603	.599	.598	.597	.596	.595
20.	.601	.599	.597	.596	.596	.596	.594
50.	.596	.595	.594	.594	.594	.594	.593
100.	.593	.592	.592	.592	.592	.592	.592

This table shows that the coefficient c decreases as the size of the orifice increases, and that for diameters less than 0.2

* Hydraulics, p. 59.

feet it decreases as the head increases. It may be presumed that the cause of this variation is due to a more perfect contraction of the jet for large heads and large orifices than for small heads and small orifices.

In applying the above coefficients to actual problems, the approximate formula

$$q = c.\tfrac{1}{4}\pi d^2 \sqrt{2gh}$$

may be used except for the values found above the horizontal lines in the last three columns. For these, if precision be required, the accurate expression for q must be employed. The error committed by using the approximate formula for the values above the horizontal lines will depend upon the ratio of d to h; as shown, in Art. 24, this error will be about two-tenths of one per cent when $h = 2d$, and about eight-tenths of one per cent when $h = d$.

Prob. 52. Find from the table the coefficient of discharge for a circular orifice of two inches diameter under a head of 1.75 feet.

Prob. 53. Compute the probable actual discharge through a circular orifice of $\tfrac{3}{4}$ inches diameter under a head of 1 foot 3 inches.

ARTICLE 39. SQUARE VERTICAL ORIFICES.

Let a square orifice whose side is d be placed with its edges truly parallel and perpendicular to a horizontal plane. Let h_1, h_2, and h be the heads of water on its upper edge, lower edge, and centre, respectively. The theoretic discharge per second is found by replacing b by d in formula (9) of Art. 22, and the actual discharge is

$$q = c.\tfrac{2}{3}d\sqrt{2g}(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}). \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

Further, as shown in Art. 22, if h be large compared with d, the discharge may be computed by the simpler formula

$$q = c.\,d^2\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25')$$

In both formulas c is the coefficient of discharge (Art. 36).

The following values of the coefficient c have been taken from a more extended table deduced by SMITH by an exhaustive discussion of experiments. They are applicable only to cases where the orifice has a sharp inner edge so that the contraction of the jet may be perfectly formed (Art. 33).

TABLE VII. COEFFICIENTS FOR SQUARE VERTICAL ORIFICES.

Head h in Feet.	Side of the Square in Feet.						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	0.598	
0.8	.652	.631	.620	.615	.605	.600	0.597
1.0	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.	.637	.619	.612	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.	.632	.616	.609	.607	.605	.604	.603
4.	.628	.614	.608	.606	.605	.603	.602
6.	.623	.612	.607	.605	.604	.603	.602
8.	.619	.610	.606	.605	.604	.603	.602
10.	.616	.608	.605	.604	.603	.602	.601
20.	.606	.604	.602	.602	.602	.601	.600
50.	.602	.601	.601	.600	.600	.599	.599
100.	.599	.598	.598	.598	.598	.598	.598

The same general laws of variation are here observed as for circular orifices, the coefficient decreasing as the head increases and as the size of the square increases. It should be noticed that the coefficients are always slightly larger than those for circles of the same diameter; this is perhaps caused by the less perfect contraction of the jet due to the corners of the square.

The horizontal lines drawn in the last three columns of the table indicate the limit $h = 4d$; so that the exact formula is to

be used for cases that fall above these lines. The error in the use of the approximate formula when $h = 3.5d$ is about one tenth of one per cent, which is probably less than the error in applying the coefficient to any given orifice in practice. For all values except those above the horizontal lines the error of the approximate formula is much less than one-tenth of one per cent.

There are few recorded experiments on large square orifices. ELLIS measured the discharge from a vertical orifice 2 feet square in an iron plate which furnishes the following results:*

For $h = 2.07$ feet, $c = 0.611$;

For $h = 3.05$ feet, $c = 0.597$;

For $h = 3.54$ feet, $c = 0.604$;

which indicate that a mean value of about 0.6 for c is all that can be safely stated for large orifices.

Prob. 54. Find from the table the coefficient of discharge for a square whose side is 3 inches when the head on its centre is 1.8 feet.

Prob. 55. Compute the probable actual discharge from a vertical orifice one foot square when the head on its upper edge is one foot. Ans. 5.85 cubic feet per second.

ARTICLE 40. RECTANGULAR VERTICAL ORIFICES.

Rectangular vertical orifices with the longest edge horizontal are frequently employed for the measurement of water. If b be the breadth, d the depth, h_1, h_2, and h the head on the upper edge, lower edge, and centre, and c the coefficient of discharge, the discharge per second is

$$q = c \cdot \tfrac{2}{3} b \sqrt{2g}(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}), \quad . \quad . \quad . \quad . \quad . \quad (26)$$

* Transactions American Society Civil Engineers, 1876, vol. v. p. 92.

or more simply, if h be greater than $4d$,

$$q = c \,.\, bd\sqrt{2gh} \,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \quad (26')$$

The following values of the coefficient c have been compiled and computed from the discussion given by FANNING.* Those above the horizontal lines are to be used in the exact formula, and those below in the approximate formula.

TABLE VIII. COEFFICIENTS FOR RECTANGULAR ORIFICES 1 FOOT WIDE.

Head h in Feet.	Depth of Orifice in Feet.						
	0.125	0.25	0.50	0.75	1.0	1.5	2.0
0.4	0.634	0.633	0.622				
0.6	.633	.633	.619	0.614			
0.8	.633	.633	.618	.612	0.608		
1.	.632	.632	.618	.612	.606	0.626	
1.5	.630	.631	.618	.611	.605	.626	0.628
2.	.629	.630	.617	.611	.605	.624	.630
2.5	.628	.628	.616	.611	.605	.616	.627
3.	.627	.627	.615	.610	.605	.614	.619
4.	.624	.624	.614	.609	.605	.612	.616
6.	.615	.615	.609	.604	.602	.606	.610
8.	.609	.607	.603	.602	.601	.602	.604
10.	.606	.603	.601	.601	.601	.601	.602
20.				.601	.601	.601	.602

This table shows that the variation of c with the head follows the same law as for circles and squares. It is also seen that for a rectangle of constant breadth the coefficient of discharge increases as its depth decreases, from which it is to be inferred that for a rectangle of constant depth the coefficient increases with the breadth, and this is confirmed by other experiments. The value of c for a rectangular orifice is seen to

* Treatise on Water Supply Engineering, p. 205.

be but slightly larger than for a square whose side is equal to the depth of the rectangle. In selecting a coefficient for use with an orifice whose size falls outside the limits of the table, it should be borne in mind that large orifices have a smaller value of c than small orifices.

A comparison of the values of c for the orifice one foot square with those in the last article shows that the two sets of coefficients disagree, these being about one per cent greater than those. This is probably due to the less precise character and smaller number of experiments from which they were deduced. Further experimental data on rectangular orifices are needed.

Prob. 56. What head is required to discharge 5 cubic feet per second through an orifice 3 inches deep and 12 inches long?

Prob. 57. What is a probable coefficient of discharge for an orifice 3 inches deep and 6 inches long; also for an orifice 1 inch deep and 6 inches long?

ARTICLE 41. THE MINER'S INCH.

The miner's inch may be roughly defined to be the quantity of water which will flow from a vertical standard orifice one inch square, when the head on the centre of the orifice is $6\frac{1}{2}$ inches. From Table VII the coefficient of discharge is seen to be about 0.623, and accordingly the actual discharge in cubic feet per second is

$$q = \frac{0.623 \times 8.02}{144}\sqrt{\frac{6.5}{12}} = 0.0255,$$

and the discharge in one minute is

$$60 \times 0.0255 = 1.53 \text{ cubic feet.}$$

The mean value of one miner's inch is therefore about 1.5 cubic feet per minute.

The actual value of the miner's inch, however, differs con-

siderably in different localities. BOWIE states that in different counties of California it ranges from 1.20 to 1.76 cubic feet per minute.* The reason for these variations is due to the fact that when water is bought for mining or irrigating purposes a much larger quantity than one miner's inch is required, and hence larger orifices than one square inch are needed. Thus, at Smartsville a vertical orifice or module 4 inches deep and 250 inches long, with a head of 7 inches above the top edge, is said to furnish 1000 miner's inches. Again, at Columbia Hill, a module 12 inches deep and 12¾ inches wide, with a head of 6 inches above the upper edge, is said to furnish 200 miner's inches. In Montana the customary method of measurement is through a vertical rectangle, one inch deep, with a head on the centre of the orifice of 4 inches, and the number of miner's inches is said to be the same as the number of linear inches in the rectangle; thus under the given head an orifice one inch deep and 60 inches long would furnish 60 miner's inches. The discharge of this is said to be about 1.25 cubic feet per minute, or 75 cubic feet per hour.

A module is an orifice which is used in selling water, and which under a constant head is to furnish a given number of miner's inches, or a given quantity per second. The sizes and proportions of modules vary greatly in different localities, but in all cases the important feature to be observed is that the head should be maintained nearly constant in order that the consumer may receive the amount of water for which he bargains and no more.

The simplest method of maintaining a constant head is by placing the module in a chamber which is provided with a gate that regulates the entrance of water from the main reservoir or canal. This gate is raised or lowered by an inspector once or twice a day so as to keep the surface of the water in the cham-

* BOWIE, Treatise on Hydraulic Mining, p. 268.

ber at a given mark. This plan though simple is costly, except in works where many modules are used, and where a daily inspection is necessary in any event, and it is not well adapted to cases where there are frequent and considerable fluctuations in the surface of the water in the feeding canal.

Numerous methods have been devised to secure a constant head by automatic appliances; for instance, the gate which admits water into the chamber may be made to rise and fall by means of a float upon the surface; the module itself may be made to decrease in size when the water rises, and to increase when it falls, by a gate or by a tapering plug which moves in and out and whose motion is controlled by a float. These self-acting contrivances, however, are liable to get out of order, and require to be inspected more or less frequently.*

The use of the miner's inch, or of a module, as a standard for selling water, may be said to have a certain advantage in simplicity, as it depends merely upon an arbitrary definition. It is, however, greatly to be desired for the sake of uniformity that water should be bought and sold by the cubic foot. Only in this way can comparisons readily be made, and the consumer be sure of obtaining exact value for his money.

Prob. 58. If a miner's inch be 1.57 cubic feet per minute, how many miner's inches will be furnished by a module 2 inches deep and 50 inches long with a head of 6 inches above the upper edge? †

ARTICLE 42. SUBMERGED ORIFICES.

It is shown in Art. 26 that the effective head which causes the flow from a submerged orifice is the difference in level between the two water surfaces. The discharge from such an

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* A cheap and simple method of maintaining a nearly constant head by means of an excess weir is described by FOOTE in the Transactions American Society of Civil Engineers for March, 1887.

‡ See BOWIE's Hydraulic Mining, page 125.

orifice, its inner edge being a sharp definite corner as in Fig. 25, has been found by experiment to be somewhat less than when the flow occurs freely, or, in other words, the values of the coefficients of discharge are smaller than those given in the preceding articles. The difference, however, is very slight for large orifices and large heads, and for orifices one inch square under six inches head is about 2 per cent.

The following table gives values of the coefficient of discharge for submerged orifices as determined by the experiments of HAMILTON SMITH, Jr. The height of the water on the exterior of the orifices varied from 0.57 to 0.73 feet above their centres.

TABLE IX. COEFFICIENTS FOR SUBMERGED ORIFICES.

Effective Head in Feet.	Size of Orifice in Feet.				
	Circle 0.05	Square 0.05	Circle 0.1	Square 0.1	Rectangle 0.05 × 0.3
0.5	0.615	0.619	0 603	0.608	0.623
1.0	.610	.614	602	.606	.622
1.5	607	.612	.600	.605	.621
2.0	.605	.610	.599	.604	.620
2.5	.603	.608	.598	.604	.619
3.0	.602	.607	.598	.604	.618
4.0	.601	.606	.598	.604	

The theoretic discharge from a submerged orifice is the same for the same effective head whatever be its distance below the lower water level. It is not likely, however, that the same coefficients of discharge would be found for deeply submerged orifices as for those submerged but slightly. Experiments in this direction from which to draw conclusions are lacking.

Prob. 59. An orifice one inch square in a gate such as shown in Fig. 7, Art. 14, is 3 feet below the higher water level and 2

feet below the lower water level. Compute the discharge in cubic feet per minute. Ans. 2.04 cubic feet.

ARTICLE 43. SUPPRESSION OF THE CONTRACTION.

When a vertical orifice has its lower edge at the bottom of the reservoir, as shown at *A* in Fig. 26, the particles of water flowing through its lower portion move in lines nearly perpendicular to the plane of the orifice, or the contraction of the jet does not form on the lower side. This is called a case of suppressed or incomplete contraction. The same thing occurs, but in a lesser degree, when the lower edge of the orifice is near the bottom as shown at *B*. In like manner, if an orifice be placed so that one of its vertical edges is at or near a side of the reservoir, as at *C*, the contraction of the jet is suppressed upon one side, and if it be placed at the lower corner of the reservoir, suppression occurs both upon one side and the lower part of the jet.

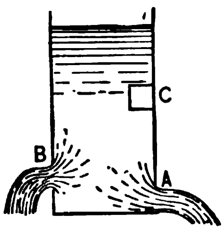


FIG. 26.

The effect of suppressing the contraction is, of course, to increase the cross-section of the jet at the place where full contraction would otherwise occur, and it is found by experiment that the discharge is likewise increased. Experiments also show that more or less suppression of the contraction will occur unless each edge of the orifice is at a distance at least equal to three times its least diameter from the sides or bottom of the reservoir.

The experiments of LESBROS and BIDONE furnish the means of estimating the increased discharge caused by suppression of the contraction. They indicate that for square orifices with contraction suppressed on one side the coefficient of discharge is increased about 3.5 per cent, and with contraction suppressed on two sides about 7.5 per cent. For a rect-

angular orifice with the contraction suppressed on the bottom edge the percentages are larger, being about 6 or 7 per cent when the length of the rectangle is four times its height, and from 8 to 12 per cent when the length is twenty times the height. The percentage of increase, moreover, varies with the head, the lowest heads giving the lowest percentages.

It is apparent that suppression of the contraction should be avoided if accurate results are desired. The experiments from which the above conclusions are deduced were made upon small orifices with heads less than 6 feet, and it is not known how they will apply to large orifices under high heads.

Prob. 60. Compute the probable discharge from a vertical orifice one foot square when the head on its upper edge is one foot, the contraction being suppressed on the lower edge.

ARTICLE 44. ORIFICES WITH ROUNDED EDGES.

If the inner edge of the orifice be rounded, as shown in Fig. 27, the contraction of the jet is modified, and the discharge is increased. With a slight degree of rounding, as at A, a partial contraction occurs; but with a more complete rounding, as at C, the particles of water issue perpendicular to the plane of the orifice and there is no contraction of the jet. If a be the area of the least cross-section of the orifice, and a' that of the jet, the coefficient of contraction (Art. 34) is

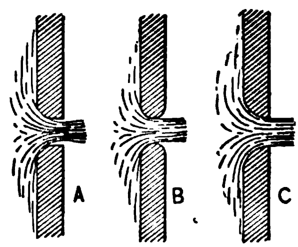


FIG. 27.

$$c' = \frac{a'}{a}.$$

For a standard square-edged orifice (Fig. 25) the mean value of c' is 0.62, but with a rounded orifice c' may have any value between 0.62 and 1.0, depending upon the degree of rounding.

The coefficient of discharge for square-edged orifices has a mean value of about 0.61; this is increased with rounded edges and may have any value between 0.61 and 1.0, although it is not probable that values greater than 0.95 can be obtained except by the most careful adjustment of the rounded edges to the exact curve of a completely contracted jet.

A rounded interior edge in an orifice is therefore always a source of error when the object of the orifice is the measurement of the discharge. It has been stated that the Romans, who used square-edged orifices as a standard for selling water, prescribed penalties for the employment by a consumer of orifices with rounded edges.

Prob. 61. If an orifice with rounded edges has a coefficient of contraction of 0.85 and a coefficient of discharge of 0.75, find the coefficient of velocity.

ARTICLE 45. THE MEASUREMENT OF WATER BY ORIFICES.

In order that water may be accurately measured by the use of orifices many precautions must be taken, some of which have already been noted, but may here be briefly recapitulated. The area of the orifice should be small compared with the size of the reservoir in order that velocity of approach may not affect the flow (Art. 25). The inner edge of the orifice must have a definite right-angled corner, and its dimensions are to be accurately determined. If the orifice be in wood, care should be taken that the inner surface be smooth, and that it be kept free from the slime which often accompanies the flow of water even when apparently clear. That no suppression of the contraction may occur, the edges of the orifice should not be nearer than three times its least dimension to a side of the reservoir.

Orifices under very low heads should be avoided, because slight variations in the head produce relatively large errors, and also because the coefficients of discharge vary more rapidly

and are probably not so well determined as for cases where the head is greater than four times the depth. For similar reasons very small orifices are not desirable. If the head be very low on an orifice, vortices will form which render any estimation of the discharge unreliable.

The measurement of the head, if required with precision, must be made with the hook gauge which is described in Art. 50. For heads greater than two or three feet the readings of an ordinary glass gauge placed upon the outside of the reservoir will usually prove sufficient, as this can be read to hundredths of a foot with accuracy. An error of 0.01 feet when the head is 3.00 feet produces an error in the computed discharge of less than two-tenths of one per cent; for, the discharges being proportional to the square roots of the heads, $\sqrt{3.01}$ divided by $\sqrt{3.00}$ equals 1.0017. For the rude measurements in connection with the miner's inch a common foot-rule will probably suffice.

The effect of temperature upon the discharge remains to be noticed; this is only appreciable with small orifices and under low heads. UNWIN found that the discharge was diminished one per cent by a rise of 144 degrees in temperature; his orifice was a circle 0.033 feet in diameter under heads ranging from 1.0 to 1.5 feet. SMITH found that the discharge was diminished one per cent by a rise of 55 degrees in temperature; his orifice was a circle 0.02 feet in diameter, under heads ranging from 0.56 to 3.2 feet. This is a further reason why small orifices and low heads are not desirable in precise measurements of discharge.

The coefficients given in the preceding tables may be supposed liable to a probable error of two or three units in the third decimal place; thus a coefficient 0.615 should really be written 0.615 ± 0.003; that is, the actual value is as likely to be between 0.612 and 0.618 as to be outside of those limits.

The probable error in computed discharges due to the coefficient is hence about one-half of one per cent. To this are added the errors due to inaccuracy of observation, so that it is thought that the probable error of careful work with standard circular orifices is at least one per cent. The computed discharges are hence liable to error in the third significant figure, so that it is useless to carry numerical results beyond four figures when based upon tabular coefficients. As a precise method of measuring small quantities of water, standard orifices take a high rank when the observations are conducted with care. With rectangular orifices the probable error is liable to be two per cent or more.

Prob. 62. What error is produced in the computed discharge if the head be read 1.38 feet when it should have been 1.385 feet?

ARTICLE 46. THE ENERGY OF THE DISCHARGE.

A jet of water flowing from an orifice possesses by virtue of its velocity a certain energy or potential work, which is always less than the theoretic energy due to the head (Art. 31). Let h be the head and W the weight of water discharged per second, then the theoretic energy per second is

$$K = Wh.$$

Let v be the actual velocity of the water at the contracted section of the jet; then the actual energy per second of the water as it passes that section is

$$k = W\frac{v^2}{2g}. \qquad (27)$$

But $\frac{v^2}{2g}$ is less than h because v is less than the theoretic velocity; or, if c_1 be the coefficient of velocity (Art. 36),

$$v = c_1\sqrt{2gh},$$

whence
$$\frac{v^2}{2g} = c_1^2 h;$$

and hence the effective energy is

$$k = c_1^2 Wh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The efficiency of the jet accordingly is

$$e = \frac{k}{K} = c_1^2,$$

which is always less than unity.

For the standard orifice with square inner edges a mean value of c_1 is 0.98. The mean effective energy of the jet at the contracted section is hence

$$k = 0.96 Wh;$$

that is, the effective energy is 96 per cent of the theoretic. For high heads c_1 is greater than 0.98, and the efficiency becomes greater than 96 per cent. It is not possible in practice to take advantage of this high efficiency, on account of the difficulty of placing the vanes of a hydraulic motor so near the orifice, and accordingly standard orifices are never used when the work of the discharge is to be utilized.

The loss of energy, or potential work, is hence about 4 per cent with the standard orifice. This is caused by the influence of the edges of the orifice which retard the velocity of the outer filaments of the jet. That these outer filaments move slower than the central ones may be seen by placing fine sand or sawdust in the water and observing that the greater part passes out of the orifice in the interior of the jet.

Prob. 63. Prove that the energy due to the velocity of the jet in the plane of the inner edge of the standard orifice is about 37 per cent of the theoretic energy. How is the remaining 63 per cent accounted for?

ARTICLE 47. DISCHARGE UNDER A DROPPING HEAD.

If a vessel or reservoir receives no inflow of water while an orifice is open, the head drops and the discharge decreases in each successive second. Let H be the head on the orifice at a certain instant, and h the head t seconds later; let A be the area of the uniform horizontal cross-section of the vessel, and a the area of the orifice. Then, as demonstrated in Art. 28, the time t is

$$t = \frac{2A}{a\sqrt{2g}}(\sqrt{H} - \sqrt{h}).$$

This is the theoretic time; to determine the actual time the coefficient of discharge must be introduced. Referring to the demonstration, it is seen that $a\sqrt{2gy}\,.\,\delta t$ is the theoretic discharge in the time δt; hence the actual discharge is $c\,.\,a\sqrt{2gy}\,\delta t$, and accordingly the above equation is to be thus modified:

$$t = \frac{2A}{ca\sqrt{2g}}(\sqrt{H} - \sqrt{h}), \quad . \quad . \quad . \quad . \quad . \quad (28)$$

which is the practical formula for the time in which the water level drops from H to h. In using this formula c may be taken from the tables in the preceding articles, an average value being selected corresponding to the average head.

Experiments have been made to determine the value of c by the help of this formula; the liquid being allowed to flow, A, a, H, h, and t being observed, whence c is computed. In this way c for mercury has been found to be about 0.62.* Only approximate mean values can be found in this manner, since c varies with the head, particularly for small orifices (Art. 38). For a large orifice the time of descent is usually so small that it cannot be noted with precision, and the friction of the liquid

* DOWNING'S Elements of Practical Hydraulics (London, 1875), p. 187.

on the sides of the vessel may also introduce an element of uncertainty. This experiment has therefore little value except as illustrating and confirming the truth of the theoretic formulas.

The discharge in one second when the head is H at the beginning of the second is found as follows: The above equation may be written in the form

$$\sqrt{H} - \frac{tca\sqrt{2g}}{2A} = \sqrt{h}.$$

By squaring both members, transposing and multiplying by A, this becomes

$$A(H-h) = tca\sqrt{2g}\left(\sqrt{H} - \frac{tca\sqrt{2g}}{4A}\right).$$

But the first term of this equation is the quantity discharged in t seconds; therefore the discharge Q for t seconds may be written

$$Q = tca\left(\sqrt{2gH} - tc\frac{ga}{2A}\right),$$

and the discharge in one second is

$$q = ca\left(\sqrt{2gH} - c\frac{ga}{2A}\right). \quad . \quad . \quad . \quad . \quad (29)$$

If $A = \infty$, this becomes $ca\sqrt{2gH}$, which should be the case, for then H would remain constant. The head at the end of one second is $h = H - \frac{q}{A}$, and at the end of t seconds is $h = H - \frac{Q}{A}$.

For example, let an orifice one foot square in a reservoir of 10 square feet section be under a head of 9 feet. The orifice having a sharp inner corner, the coefficient of discharge from Table VII is 0.602. Then the discharge in one second is 13.9 cubic feet, and the head drops to 7.61 feet. The discharge in

the second second is 12.7 cubic feet, and the head drops to 6.34 feet, and so on. The time required to discharge a given quantity may be found from the formula for Q by solving for t, or preferably from the first formula, h being computed from the given data.

It is shown in Art. 25 that if the head be maintained constant, the theoretic velocity of flow is

$$v = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}}.$$

Hence the actual discharge may be written

$$q = ca\sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}}.$$

This furnishes another method of computing the discharge under a dropping or rising head, when the heads are determined by observations at uniform intervals, as is usually the case in practice. The discharge per second may be computed from this formula, or, if the orifice be small, from,

$$q = c \, . \, a\sqrt{2gh},$$

taking h as constant during one second. By computing successive values of q corresponding to successive observed values of h, the variation in the discharge is thus found. It is not advisable, however, to allow the head to drop or rise rapidly in hydraulic measurements. When such cases occur h should be observed at least every half-minute; the values of q computed from these readings should be plotted on cross-section paper, and the curve drawn through the points then shows the law of variation, and intermediate values can be obtained without the necessity of computation.

Prob. 64. Find the time required to discharge 480 gallons from an orifice 2 inches in diameter at 8 feet below the water level in a tank which is 4 × 4 feet in cross-section.

ARTICLE 48. EMPTYING AND FILLING A CANAL LOCK.

A canal lock is emptied by opening one or more orifices in the lower gates. Let a be their area, and H the head of water on them when the lock is full; let A be the area of the horizontal cross-section of the lock. Then in the formula of the last article, $h = 0$, and the time of emptying the lock is

$$t = \frac{2A\sqrt{H}}{ca\sqrt{2g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If the discharge be free into the air, H is the distance from the centre of the orifice to the level of the water in the lock when filled; but if, as is usually the case, the orifices be below the level of the water in the tail bay, H is the difference in height between the two water levels. The tail bay is regarded as so large compared with the lock that its water level remains constant.

For example, let it be required to find the time of emptying a canal lock 80 feet long and 20 feet wide through two orifices, each of 4 square feet area, the head upon which is 16 feet when the lock is filled. Using for c the value 0.6 for orifices with square inner edges, the formula gives

$$t = \frac{2 \times 80 \times 20 \times 4}{0.6 \times 8 \times 8.02} = 333 \text{ seconds} = 5\tfrac{1}{2} \text{ minutes.}$$

If, however, the circumstances be such that c is 0.8, the time is about 250 seconds, or $4\frac{1}{6}$ minutes. It is therefore seen that it is important to arrange the orifices of discharge in canal locks with rounded inner edges so that c may be as near unity as

possible, in order both to make the orifices with their gates as small as practicable, and to diminish the time of emptying the lock.

The filling of the lock is the reverse operation. Here the water in the head bay remains at a constant level, and the discharge through the orifices in the upper gates occurs at first quickly, diminishing with the rising head in the lock. Let H be the effective head on the orifices when the lock is empty,

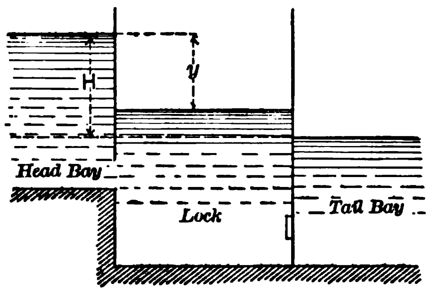


FIG. 28.

and y the effective head at any time t after the beginning of the discharge into the lock. The area of the section of the lock being A, the quantity $A\delta y$ is discharged in the time δt, and this is equal to $ca\sqrt{2gy}\,\delta t$, if a be the area of the orifices and c the coefficient of discharge. Hence

$$\delta t = \frac{A\delta y}{ca\sqrt{2gy}};$$

and by integration between the limits 0 and H,

$$t = \frac{2A\sqrt{H}}{ca\sqrt{2g}},$$

which is the same as the formula for the time of emptying the lock. The times of filling and emptying a lock are therefore

equal if the orifices for inflow and outflow are of the same dimensions and under the same heads. Usually the upper orifice is under a less head than the lower, and hence its area must be larger if the time of filling is required to be the same as that of emptying. The area a for any case is found by the equation

$$a = \frac{2A\sqrt{H}}{ct\sqrt{2g}},$$

in which A, H, and t are given, and c is determined from the evidence presented in the preceding pages.

Prob. 65. A lock has a horizontal cross-section of 1800 square feet, and the lift H is 12 feet. Find the size of the orifices for emptying it in 3 minutes when the coefficient of discharge is 0.7.

Ans. $a = 12.3$ square feet.

CHAPTER V.

FLOW OF WATER OVER WEIRS.

ARTICLE 49. DESCRIPTION OF A WEIR.

A weir is a notch in the top of the vertical side of a vessel or reservoir through which water flows. The notch is generally rectangular, and the word weir will be used to designate a rectangular notch unless otherwise specified, the lower edge of the rectangle being truly horizontal, and its sides vertical. The lower edge of the rectangle is called the "crest" of the weir.

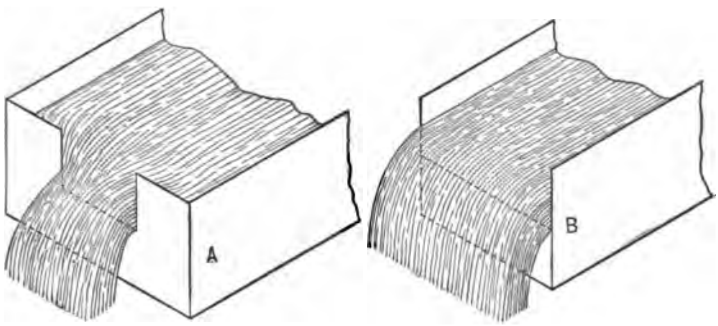


FIG. 29.

In Fig. 29 are shown the outlines of two kinds of weirs, *A* being the more usual form where the vertical edges of the notch are sufficiently removed from the sides of the reservoir or feeding canal, so that the sides of the stream may be fully contracted; this is called a weir with end contractions. In the form at *B*, the edges of the notch are coincident with the sides of the feeding canal, so that the filaments of water along the sides pass over without being deflected from the vertical planes in which they move; this is called a weir without end contractions, or with end contractions suppressed.

It is necessary in order to make accurate measurements of discharge by a weir that the same precaution should be taken

as for orifices (Art. 34), namely, that the inner edge of the notch shall be a definite angular corner so that the water in flowing out may touch the crest only in a line, thus insuring complete contraction. In precise observations a thin metal plate will be used for a crest as seen in Fig. 30, while in common work it may be sufficient to have the crest formed by a plank of smooth hard wood with its inner corner cut to a sharp right angle and its outer edge bevelled. The vertical edges of the weir should be made in the same manner for weirs with end contractions, while for those without end contractions the sides of the feeding canal should be smooth and be prolonged a slight distance beyond the crest. It is also necessary to observe the same precautions as for orifices to prevent the suppression of the contraction (Art. 43), namely, that the distance from the crest of the weir to the bottom of the feeding canal, or reservoir, should be greater than three times the head of water on the crest. For a weir with end contractions a similar distance should exist between the vertical edges of the weir and the sides of the feeding canal.

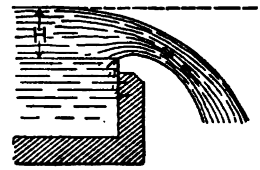


FIG. 30.

The head of water H upon the crest of a weir is usually much less than the breadth of the crest, b. The value of H should not be less than 0.1 foot, and it rarely exceeds 1.5 feet. The least value of b in practice is about 0.5 feet, and it does not often exceed 20 feet. Weirs are extensively used for measuring the discharge of streams, and for determining the quantity of water supplied to hydraulic motors; the practical importance of the subject is so great that numerous experiments have been made to ascertain the laws of flow, and the coefficients of discharge.

Prob. 66. If a feeding canal three feet wide discharges 12 cubic feet per second when the water is 2 feet deep, what is the mean velocity of flow?

FIG. 31.

ARTICLE 50. THE HOOK GAUGE.

As the head on the crest of a weir is low it must be determined with precision in order to avoid error in the computed discharge (Art. 45). The hook gauge, invented by BOYDEN about 1840, consists of a rod sliding vertically in fixed supports, the amount of vertical motion being determined by the readings of a vernier. The vernier can be set to read 0.000 when the sharp point of the hook is on the same level as the crest of the weir; when the water is flowing over the crest, the rod is raised by the slow-motion screw until the point of the hook is at the water level. Before the point pierces the surface or skin of the water, a pimple or protuberance is seen to rise above it due to capillary action; the hook is then depressed until this pimple is barely perceptible, when the point is at the true water level. The head of water on the crest is then indicated by the reading of the scale and vernier. The best hook gauges are made to read to ten-thousandths of a foot, and it has been stated that an experienced observer can in a favorable light detect differences in level as small as 0.0002 feet. The surface of water at the hook must be perfectly quiet, and hence a box without a bottom or with openings to admit the water is often placed around it. Fig. 31 shows the hook gauge as arranged by EMERSON.*

* EMERSON's Hydrodynamics (Springfield, Mass., 1881), p. 56.

A cheaper form of hook gauge, and one sufficiently precise in some classes of work, can be made by screwing a hook into the foot of a levelling rod. The back part of the rod is then held in a vertical position by two clamps on fixed supports, while the front part is free to slide. It is easy to arrange a slow-motion movement so that the point of the hook may be precisely placed at the water level. The reading of the vernier is determined when the point of the hook is on the same level as the crest of the weir, and by subtracting from this the subsequent readings the heads of water are known. A New York levelling rod reading to thousandths of a foot is to be preferred.

The greatest error of a hook gauge is thought to be in setting it for the level of the crest. In the larger forms of hooks this may be done by taking elevations of the crest, and of the point of the hook by means of an engineer's level and a light rod. With smaller hooks it may be done by having a stiff permanent hook the elevation of whose point with respect to the crest is determined by precise levels; the water is then allowed to rise slowly until it reaches the point of this stiff hook, when readings of the vernier of the lighter hook are taken. Another method is to allow a small depth of water to flow over the crest and to take readings of the hook, while at the same time the depth on the crest is measured by a finely graduated scale. Still another way is to allow the water to rise slowly, and to set the hook at the water level when the first filaments pass over the crest; this method is not a very precise one on account of capillary attraction along the crest. As the error in setting the hook is a constant one which affects all the subsequent observations, especial care should be taken to reduce it to a minimum by taking a number of observations from which to obtain a precise mean result.

In rough gaugings of streams the precision of a hook gauge is often not required, and the heads may be determined by

simpler methods. For example, a post may be set with its top on the same level as the crest of the weir, and the depth of water over the top of the post be measured by a scale graduated to tenths and hundredths of a foot, the thousandths being either estimated or omitted entirely.

The head H on the crest of the weir is in all cases to be measured several feet up stream from the crest, as indicated in Fig. 30. This is necessary because of the curve taken by the surface of the water in approaching the weir. The distance to which this curve extends back from the weir depends upon many circumstances (Art. 59), but it is considered that perfectly level water will be found at 2 or 3 feet distance back for small weirs, and at 6 or 8 feet for very large weirs. It is desirable that the hook should be placed at least one foot from the sides of the feeding canal, if possible. As this is apt to render the position of the observer uncomfortable, some experimenters have placed the hook in a pail at a few feet distance from the canal, the water being led to the pail by a pipe: this pipe should enter the feeding canal several feet above the crest, and the water should enter it, not at its end, but through a number of holes drilled at intervals along its circumference.

Prob. 67. Show by using formula (9)′ of Art. 22 that an error of about one-half of one per cent results in the discharge if an error of 0.001 feet be made in reading the head when $H = 0.3$ feet.

ARTICLE 51. FORMULAS FOR THE DISCHARGE.

The theoretic discharge through a rectangular notch or weir was found in Art. 22 to be

$$Q = \tfrac{2}{3}\sqrt{2g}\,.\,bH^{\frac{3}{2}},$$

in which b is the breadth of the notch, commonly called the length of the weir, and H the depth of water on the lower

edge. It might be inferred that this depth is that in the plane of the weir; but as the deduction of the formula supposes nothing regarding the fall due to the surface curve, and regards the velocity at any point above the crest as due to the head upon that point below the free water surface, it seems that H should be measured with reference to that surface, as is actually done by the hook gauge. The above formula then gives the theoretic discharge per second, provided that there be no velocity at the point where H is measured, which can only be the case when the area of the weir opening is very small compared to that of the cross-section of the feeding canal. This condition would be fulfilled for a rectangular notch placed at the side of a large pond.

When there is an appreciable velocity of approach of the water at the point where H is measured by the hook gauge, the above formula must be modified. Let v be the mean velocity in the feeding canal at this section; this velocity may be regarded as due to a fall, h, from the surface of still water at some distance up stream from the hook, as shown in Fig. 32. Now the true head on the crest of the weir is $H+h$, as this would have been the reading of the hook gauge had it been placed where the water had no velocity. Accordingly the theoretic discharge is

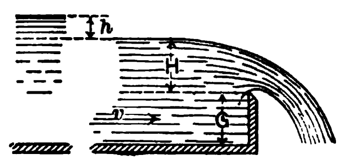


FIG. 32.

$$Q = \tfrac{2}{3}\sqrt{2g}\,.\,b(H+h)^{\frac{3}{2}},$$

in which H is read by the hook and h is determined from the mean velocity v.

The actual discharge per second is always less than the theoretic discharge, due to the contraction of the stream and the resistances of the edges of the weir. To take account of

these a coefficient is applied to the theoretic formulas in the same manner as for orifices; these coefficients being determined by experiment, the formulas may then be used for computing the actual discharge. It has also been proposed by SMITH to modify the velocity-head h, owing to the fact that the velocity of approach is not constant throughout the section, but greater near the surface than near the bottom, as in streams (Art. 107). Accordingly the following may be written as an expression for the actual discharge:

$$q = c \cdot \tfrac{2}{3} \sqrt{2g} \cdot b(H + nh)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad (31)$$

in which c is the coefficient of discharge whose value is always less than unity, and n is a number which lies between 1.0 and 1.5.*

The above formulas are not in all respects perfectly satisfactory, and indeed many others have been proposed. The actual discharge differs, however, so much from the theoretical that the final dependence must be upon the coefficients deduced from experiment, and hence any fairly reasonable formula may be used within the limits for which its coefficients have been established. In spite of the objections which may be raised against all forms of formulas, the fact remains that the measurement of water by weirs is one of the most convenient methods, and probably the most precise method, unless the quantity is so small as to pass through a circular orifice less than one foot in diameter. With proper precautions the probable error in measurements of discharge by weirs should be less than two or three per cent.

Prob. 68. Find the velocity-head h when the mean velocity of approach is 20 feet per minute.

* SMITH's Hydraulics, p. 33.

ARTICLE 52. VELOCITY OF APPROACH.

The velocity-head h, which produces the mean velocity of approach v is (Art. 20)

$$h = \frac{v^2}{2g} = 0.01555v^2.$$

Accordingly to obtain h the value of v must be determined. One way of doing this is to observe the time of passage of a float through a given distance; but this is not a precise method. The usual method is to compute v from an approximate value of the discharge, which is first computed by regarding v, and hence h, as zero. This determination is rendered possible by the fact that v is usually small, and hence that h is quite small as compared with H.

Let B be the breadth of the cross-section of the feeding canal at the place where the readings of the hook are taken, and let G be its depth below the crest (Fig. 32). The area of that cross-section then is

$$A = B(G+H).$$

The mean velocity in this section now is

$$v = \frac{q'}{A},$$

in which q' is found from the formula

$$q' = c\,\tfrac{2}{3}\sqrt{2g}\,.\,bH^{\frac{3}{2}}.$$

This value of q' is an approximation to the actual discharge; from it v is found, and then h, after which the discharge q can be computed. If thought necessary, h may be recomputed by using q instead of q'; but this will rarely be necessary.

For example, the small weir with end contractions used in the hydraulic laboratory of Lehigh University has $B = 7.82$

feet and $G = 2.5$ feet. The length of the weir b is adjustable according to the quantity of water delivered by the stream. On April 10, 1888, the value of b was 1.330 feet, and values of H ranged from 0.429 to 0.388 feet. It is required to find the velocity v and the velocity-head h, when $H = 0.429$ feet. Here the coefficient c is 0.602 (Art. 53), hence the approximate discharge per second is

$$q' = 0.602 \times \tfrac{2}{3} \times 8.02 \times 1.33 \times 0.429^{\frac{3}{2}},$$

or $$q' = 1.203 \text{ cubic feet per second.}$$

The mean velocity of approach then is

$$v = \frac{1.203}{(2.5 + 0.4)7.82} = 0.053 \text{ feet per second,}$$

from which the velocity-head h is

$$h = \frac{0.053^2}{64.32} = 0.00004 \text{ feet.}$$

This is too small to be regarded, since the hook gauge used determines the heads only to thousandths of a foot.

The velocity-head h may be directly expressed in terms of the discharge by substituting for v its value $\frac{q}{A}$; thus:

$$h = 0.01555\left(\frac{q}{A}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In general, this expression will be found the most convenient one for computing the value of the head corresponding to the velocity of approach.

With a weir opening of given size under a given head H, the velocity of approach is less the greater the area of the section of the feeding canal, and it is desirable in building a weir to make this area large so that the velocity v may be small.

For large weirs, and particularly for those without end contractions, v is sometimes as large as one foot per second, giving $h = 0.0155$ feet, and these should be regarded as the highest values allowable if precision of measurement is required.

Prob. 69. FTELEY and STEARNS' large suppressed weir had the following dimensions: $b = B = 18.996$ feet, $G = 6.55$ feet, and the greatest measured head was 1.6038 feet. Taking $c = 0.622$, compute the velocity of approach and its velocity-head.

ARTICLE 53. WEIRS WITH END CONTRACTIONS.

Let b be the breadth of the notch or length of the weir, H the head above the crest measured by the hook gauge, and c an experimental coefficient. Then if there be no velocity of approach the discharge per second is

$$q = c \cdot \tfrac{2}{3} \sqrt{2g} \cdot bH^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

But if the mean velocity of approach at the section where the hook is placed be v, let h be the head which would produce this velocity. Then the discharge per second is

$$q = c \cdot \tfrac{2}{3} \sqrt{2g} \cdot b(H + 1.4h)^{\frac{3}{2}} . \quad . \quad . \quad . \quad . \quad (33)'$$

The quantity $H + 1.4h$ is called the effective head on the crest, and, as shown in the last article, h is usually small compared with H.

The following table contains values of the coefficient of discharge c as deduced by HAMILTON SMITH, Jr.,* from a discussion of the experiments made by LESBROS, FRANCIS, FTELEY and STEARNS, and others. In these experiments q is determined by actual measurement in a vessel of large size, and the other quantities being observed c is computed. Values of c for different lengths of weir and for different heads are thus

* Hydraulics (London and New York, 1884), p. 132.

obtained, which being plotted enable mean curves to be drawn, from which intermediate values are taken. The heads in the first column are the effective heads $H + 1.4h$; but as h is small, little error can result in using H as the argument with which to enter the table in selecting a coefficient.

TABLE X. COEFFICIENTS FOR CONTRACTED WEIRS.

Effective Head in Feet.	Length of Weir in Feet.						
	0.66	1	2	3	5	10	19
0.1	0.632	0.639	0.646	0.652	0.653	0.655	0.656
0.15	.619	.625	.634	.638	.640	.641	.642
0.2	.611	.618	.626	.630	.631	.633	.634
0.25	.605	.612	.621	.624	.626	.628	.629
0.3	.601	.608	.616	.619	.621	.624	.625
0.4	.595	.601	.609	.613	.615	.618	.620
0.5	.590	.596	.605	.608	.611	.615	.617
0.6	.587	.593	.601	.605	.608	.613	.615
0.7		.590	.598	.603	.606	.612	.614
0.8			.595	.600	.604	.611	.613
0.9			.592	.598	.603	.609	.612
1.0			.590	.595	.601	.608	.611
1.2			.585	.591	.597	.605	.610
1.4			.580	.587	.594	.602	.609
1.6				.582	.591	.600	.607

It is seen from the table that the coefficient increases with the length of the weir, which is due to the influence of the end contractions being independent of the length. The coefficient also increases as the head on the crest diminishes. The table also shows that the greatest variation in the coefficients occurs under small heads, which are hence to be avoided in order to secure accurate measurements of discharge.

Interpolation may be made in this table for heads and lengths of weirs intermediate between the values given, regard-

ing the coefficients as varying uniformly; but it will be better in any actual case to diagram the coefficients on cross-section paper, from which the interpolation can be made more easily and accurately.

As an example of the use of the formula and table, let it be required to find the discharge per second over a weir 4 feet long when the head H is 0.457 feet, there being no velocity of approach. From the table the coefficient of discharge is 0.614 for $H = 0.4$ and 0.6095 for $H = 0.5$, which gives about 0.612 when $H = 0.457$. Then the discharge per second is

$$q = 0.612 \times \tfrac{2}{3} \times 8.02 \times 4 \times 0.457^{\frac{3}{2}} = 4.04 \text{ cubic feet}$$

If the width of the feeding canal be 7 feet, and its depth below the crest be 1.5 feet, the velocity-head is

$$h = 0.01555\left(\frac{4.04}{7 \times 1.96}\right)^2 = 0.00134 \text{ feet.}$$

The effective head now becomes

$$H + 1.4h = 0.459 \text{ feet,}$$

and the discharge per second is

$$q = 0.612 \times \tfrac{2}{3} \times 8.02 \times 4 \times 0.459^{\frac{3}{2}} = 4.07 \text{ cubic feet.}$$

It is to be observed that the reliability of these computed discharges depends upon the precision of the observed quantities and upon the coefficient c; this is probably liable to an error of one or two units in the third decimal place, which is equivalent to a probable error of about three-tenths of one per cent. On the whole, regarding the inaccuracies of observation, a probable error of one per cent should at least be inferred, so that the value $q = 4.07$ cubic feet per second should strictly be written,

$$q = 4.07 \pm 0.04;$$

that is to say, the discharge per second has 4.07 cubic feet for its most probable value, and it is as likely to be between the values 4.03 and 4.11 as to be outside of those limits.

Prob. 70. Compute the discharges per second through a weir whose length is 2.5 feet, width of feeding canal 6 feet, depth below crest 1.6 feet when the heads on the crest are 0.314, 0.315, and 0.316 feet.

Prob. 71. Compute the coefficient of discharge for the following experiment by FRANCIS, in which q was found by actual measurement in a large tank: $b = 9.997$ feet, $B = 13.96$ feet, $G = 4.19$ feet, $H = 1.5243$ feet, $2g = 64.3236$ and $q = 61.282$ cubic feet per second. Ans. $c = 0.602$.

ARTICLE 54. WEIRS WITHOUT END CONTRACTIONS.

For weirs without end contractions, or suppressed weirs, when there is no velocity of approach, the discharge per second is

$$q = c \cdot \tfrac{2}{3}\sqrt{2g} \cdot bH^{\frac{3}{2}}; \quad \ldots \ldots \quad (34)$$

and when there is velocity of approach,

$$q = c \cdot \tfrac{2}{3}\sqrt{2g} \cdot b(H + 1\tfrac{1}{3}h)^{\frac{3}{2}}. \quad \ldots \ldots \quad (34)'$$

Here the notation is the same as in the last article, and c is to be taken from the following table, which gives the coefficients of discharge as deduced by SMITH.

It is seen that the coefficients for suppressed weirs are greater than for those with end contractions: this of course should be the case, as contractions diminish the discharge. They decrease with the length of the weir, while those for contracted weirs increase with the length. Their greatest variation occurs under low heads, where they rapidly increase as the head diminishes. It should be observed that these coefficients are not reliable for lengths of weirs under 4 feet,

owing to the few experiments which have been made for short weirs. Hence, for small quantities of water, weirs with

TABLE XI. COEFFICIENTS FOR SUPPRESSED WEIRS.

Effective Head in Feet.	Length of Weir in Feet.						
	19	10	7	5	4	3	2
0.1	0.657	0.658	0.658	0.659			
0.15	.643	.644	.645	.645	0.647	0.649	0.652
0.2	.635	.637	.637	.638	.641	.642	.645
0.25	.630	.632	.633	.634	.636	.638	.641
0.3	.626	.628	.629	.631	.633	.636	.639
0.4	.621	.623	.625	.628	.630	.633	.636
0.5	.619	.621	.624	.627	.630	.633	.637
0.6	.618	.620	.623	.627	.630	.634	.638
0.7	.618	.620	.624	.628	.631	.635	.640
0.8	.618	.621	.625	.629	.633	.637	.643
0.9	.619	.622	.627	.631	.635	.639	.645
1.0	.619	.624	.628	.633	.637	.641	.648
1.2	.620	.626	.632	.636	.641	.646	
1.4	.622	.629	.634	.640	.644		
1.6	.623	.631	.637	.642	.647		

end contractions should be built in preference to suppressed weirs. For a weir of infinite length it would be immaterial whether end contractions existed or not; hence for such a case the coefficients lie between the values for the 19-foot weir in Table X. and those for the 19-foot weir in the table here given.

For a numerical illustration the same data as in the example of the last article will be used, namely, $b = 4$ feet, $G = 1.5$ feet, and $H = 0.457$ feet. The coefficient from the

table is 0.630; then for no velocity of approach the discharge per second is

$$q = 0.630 \times \tfrac{2}{3} \times 8.02 \times 4 \times 0.457^{\frac{3}{2}} = 4.16 \text{ cubic feet.}$$

Here the width B would probably be also 4 feet; the head corresponding to the velocity of approach then is

$$h = 0.01555\left(\frac{4.16}{4 \times 1.96}\right)^2 = 0.0044 \text{ feet,}$$

and the effective head is

$$H + 1\tfrac{1}{3}h = 0.463 \text{ feet,}$$

from which the discharge per second is

$$q = 0.630 \times \tfrac{2}{3} \times 8.02 \times 4 \times 0.463^{\frac{3}{2}} = 4.24 \text{ cubic feet.}$$

This shows that the velocity of approach exerts a greater influence upon the discharge than in the case of a weir with end contractions.

Prob. 72. Compute the discharge per second over a weir without end contractions when $b = 9.995$ feet, $H = 0.7955$ feet, $G = 4.6$ feet. Ans. $q = 23.7$ cubic feet per second.

ARTICLE 55. FRANCIS' FORMULAS.

The formulas most extensively used for computing the flow through weirs are those established by FRANCIS in 1854* from the discussion of his numerous and carefully conducted experiments, but as they are stated without tabular coefficients they are to be regarded as giving only mean approximate results. The experiments were made on large weirs, most of them 10 feet long, and with heads ranging from 0.4 to 1.6 feet, so that the formulas apply particularly to such, rather than to short weirs and low heads. The length b and the head H being

* Lowell Hydraulic Experiments (4th edition, New York, 1883), p. 133.

expressed in feet, the discharge per second, when there is no velocity of approach, is, for weirs without end contractions, or suppressed weirs,

$$q = 3.33bH^{\frac{3}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and for weirs with end contractions,

$$q = 3.33\,(b-0.2H)H^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Here it is regarded that the effect of each end contraction is to diminish the effective length of the weir by $0.1H$.

FRANCIS' method of correcting for velocity of approach differs from that of SMITH, and is the same as that explained in Art. 25. The head h causing the velocity of approach is computed in the usual way, and then the formulas are written, for weirs without end contractions,

$$q = 3.33b[(H+h)^{\frac{3}{2}}-h^{\frac{3}{2}}]; \quad . \quad . \quad . \quad . \quad . \quad (35)'$$

and for weirs with end contractions,

$$q = 3.33(b-0.2H)[(H+h)^{\frac{3}{2}}-h^{\frac{3}{2}}]. \quad . \quad . \quad . \quad (36)'$$

It is necessary that this method of introducing the velocity of approach should be strictly observed, since the mean number 3.33 was deduced for this form of expression.

It is seen that the number 3.33 is $c \, . \, \frac{2}{3}\sqrt{2g}$, where c is the true coefficient of discharge. The 88 experiments from which this mean value was deduced show that the coefficient 3.33 actually ranged from 3.30 to 3.36, so that by its use an error of one per cent in the computed discharge may occur. When such an error is of no importance the formula may be safely used for weirs longer than 4 feet and heads greater than 0.4 feet.

Prob. 73. Find by FRANCIS' formulas the discharge when $B = 7$ feet, $b = 4$ feet, $H = 0.457$ feet, and $G = 1.5$ feet, the weir being one with end contractions.

ARTICLE 56. SUBMERGED WEIRS.

When the water on the down-stream side of the weir is allowed to rise higher than the level of the crest the weir is said to be submerged. In such cases an entire change of condition results, and the preceding formulas are inapplicable. Let H be the head above the crest measured up stream from the weir by the hook gauge in the usual manner, and let H' be the head above the crest of the water down stream from the weir measured by a second hook gauge. If H be constant, the discharge is uninfluenced until the lower water rises to the level of the crest, provided that free access of air is allowed beneath the descending sheet of water. But as soon as it rises slightly above the crest so that H' has small values, the contraction is suppressed and the discharge hence increased. As H' increases, however, the discharge diminishes until it becomes zero when H' equals H. Submerged weirs cannot be relied upon to give precise measurements of discharge on account of the lack of experimental knowledge regarding them, and should hence always be avoided if possible.

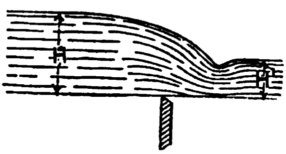
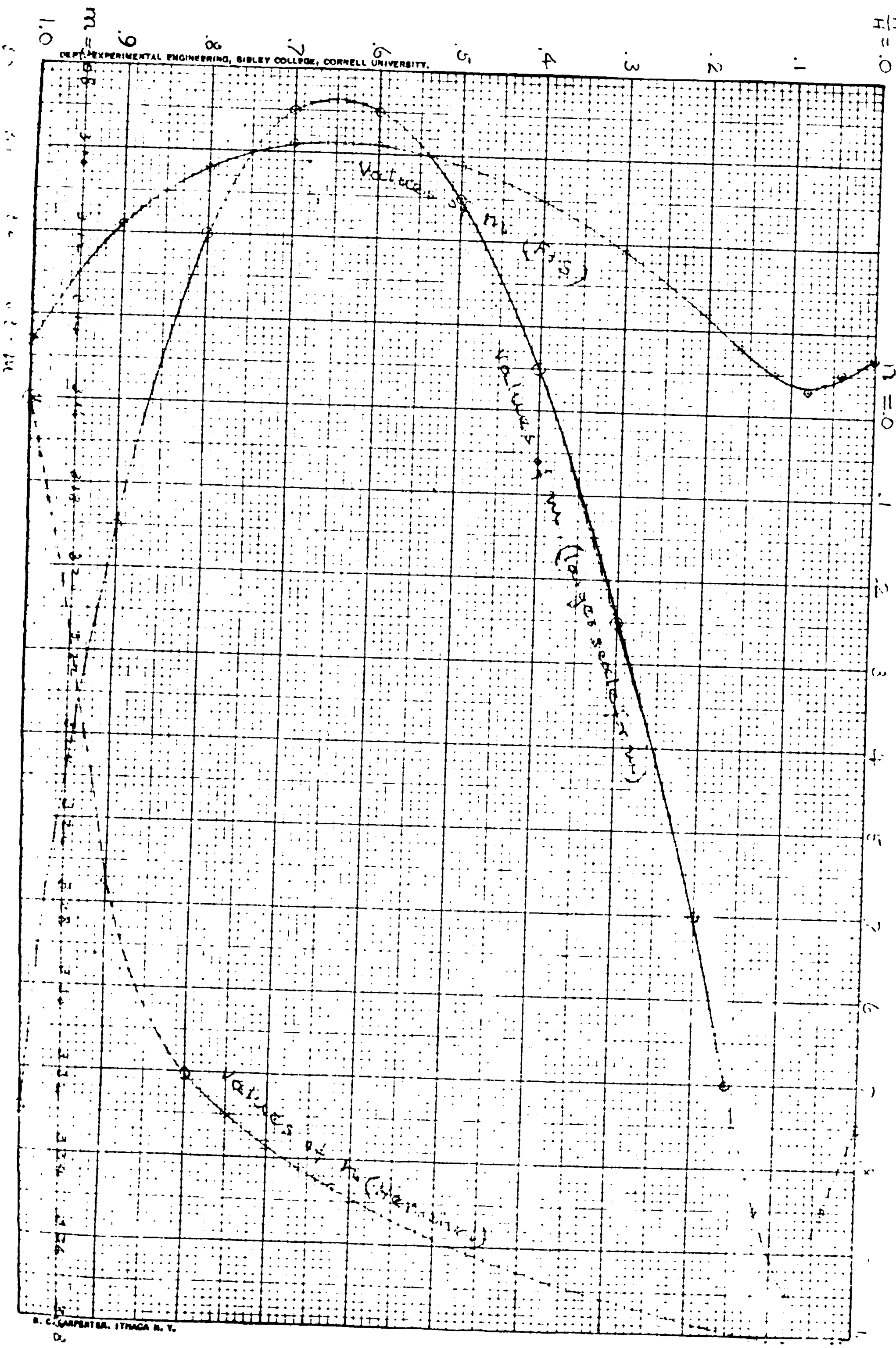


FIG. 33.

The following method for estimating the discharge over submerged weirs without end contractions is taken from the discussion given by HERSCHEL* of the experiments made by FRANCIS and by FTELEY and STEARNS. The observed head H is first multiplied by a number n, which depends upon the ratio of H' to H, and then the discharge is to be found by the formula

$$q = 3.33b(nH)^{\frac{3}{2}}.$$

* Transactions American Society of Civil Engineers, 1885, vol. xiv. p. 194.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The values of n are given in the following table:

TABLE XII. SUBMERGED WEIRS.

$\frac{H'}{H}$	n	$\frac{H'}{H}$	n	$\frac{H'}{H}$	n	$\frac{H'}{H}$	n
0.00	1.000	0.18	0.989	0.38	0.935	0.58	0.856
.01	1.004	.20	0.985	.40	0.929	.60	0.846
.02	1.006	.22	0.980	.42	0.922	.62	0.836
.04	1.007	.24	0.975	.44	0.915	.64	0.824
.06	1.007	.26	0.970	.46	0.908	.66	0.813
.08	1.006	.28	0.964	.48	0.900	.70	0.787
.10	1.005	.30	0.959	.50	0.892	.75	0.750
.12	1.002	.32	0.953	.52	0.884	.80	0.703
.14	0.998	.34	0.947	.54	0.875	.90	0.574
.16	0.994	.36	0.941	.56	0.866	1.00	0.000

The numbers in this table are liable to a probable error of about one unit in the second decimal place when H' is less than $0.2H$, and to greater errors in the remainder of the table, those values of n less than 0.70 being in particular uncertain. This discussion shows that H' may be nearly one-fifth of H without affecting the discharge more than two per cent.

A rational formula for the discharge over submerged weirs may be deduced in the following manner. The theoretic discharge may be regarded as composed of two portions, one through the upper part $H - H'$, and the other through the lower part H'. The portion through the upper part is given by the usual weir formula, $H - H'$ being the head, or

$$Q_1 = \tfrac{2}{3} \sqrt{2g}\, b (H - H')^{\frac{3}{2}};$$

and that through the lower part is given by the formula for a submerged orifice (Art. 42), in which b is the breadth, H' the height, and $H - H'$ the effective head, or

$$Q_2 = bH' \sqrt{2g(H - H')}.$$

The addition of these gives the total theoretic discharge,

$$Q = \tfrac{2}{3}\sqrt{2g}\, b(H - H')^{\frac{3}{2}} + \sqrt{2g}\, bH'(H - H')^{\frac{1}{2}}.$$

This may be put into the more convenient form,

$$Q = \tfrac{2}{3}\sqrt{2g}\, b(H + \tfrac{1}{2}H')(H - H')^{\frac{1}{2}}.$$

The actual discharge per second may now be written,

$$q_1 = c\,.\,\tfrac{2}{3}\sqrt{2g}\, b(H + \tfrac{1}{2}H')(H - H')^{\frac{1}{2}}; \quad . \quad . \quad (37)$$

in which c is the coefficient of discharge.

FTELEY and STEARNS adopt the above formula for the discharge, or placing m for $c\,.\,\tfrac{2}{3}\sqrt{2g}$, they write,*

$$q_1 = mb(H + \tfrac{1}{2}H')(H - H')^{\frac{1}{2}}; \quad . \quad . \quad . \quad (37)'$$

and from their experiments deduce the following values of m:

For $\frac{H'}{H}$ =	0.00	0.04	0.08	0.12	0.16	0.2	0.3
m =	3.33	3.35	3.37	3.35	3.32	3.28	3.21
For $\frac{H'}{H}$ =	0.4	0.5	0.6	0.7	0.8	0.9	1.0
m =	3.15	3.11	3.09	3.09	3.12	3.19	3.33

These are for suppressed weirs; for contracted weirs few or no experiments are on record.

In what has thus far been said velocity of approach has not been considered. This may be taken into account in the usual way by determining the velocity-head h, and thus correcting H. In strictness the velocity of departure in the tail bay below

* Transactions American Society Civil Engineers, 1883, vol. xii. p. 103.

Merriman's Hydraulics, 10th Ed., page 344
Backwater due to Bridge Piers — General Case

$$d = \frac{v^2}{2g}\left[\left(\frac{A}{ca}\right)^2 - \left(\frac{A}{A_1}\right)^2\right], \quad 2g = 64.3236, \quad 1/2g = .0155464$$

$$c = 0.75 + 0.35(a/A) - 0.10(a/A)^2$$

d = rise due to obstruction
a = original uncontracted area (of stream)
A = the contracted area (of stream)
A_1 = A + Bd, where B = surface width (of stream)
Note: This formula may apply to a long submerged weir or to some other obstruction in a river.

$\left(\frac{a}{A}\right)$	$.35\left(\frac{a}{A}\right)$	$.10\left(\frac{a}{A}\right)^2$	c	
1.00	0.35000	.10000	1.00000	
.99	.34650	.09801	.99849	151
.98	.34300	.09604	.99696	153
.97	.33950	.09409	.99541	155
.96	.33600	.09216	.99384	157
0.95	0.33250	0.09025	0.99225	159
.94	.32900	.08836	.99064	161
.93	.32550	.08649	.98901	163
.92	.32200	.08464	.98736	165
.91	.31850	.08281	.98569	167
0.90	0.31500	0.08100	0.98400	169
.89	.31150	.07921	.98229	171
.88	.30800	.07744	.98056	173
.87	.30450	.07569	.97881	175
.86	.30100	.07396	.97704	177
0.85	0.29750	0.07225	0.97525	179
.84	.29400	.07056	.97344	181
.83	.29050	.06889	.97161	183
.82	.28700	.06724	.96976	185
.81	.28350	.06561	.96789	187
0.80	0.28000	0.06400	0.96600	189
.79	.27650	.06241	.96409	191
.78	.27300	.06084	.96216	193
.77	.26950	.05929	.96021	195
.76	.26600	.05776	.95824	197
0.75	0.26250	0.05625	0.95625	199

Over

THE SANITARY DISTRICT OF CHICAGO
ENGINEERING DEPARTMENT

Subject ______ Computation ______

Data ______ Page ______

Computed by ______ Checked by ______ Date ______ 191_

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68	94176	215	33	85461	285
67	93961	217	32	85176	287
66	93744	219	31	84889	289
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64	93304	223	29	84309	293
63	93081	225	28	84016	295
62	92856	227	27	83721	297
61	92629	229	26	83424	299
0.60	0.92400	231	0.25	0.83125	301
59	92169	233	24	82824	303
58	91936	235	23	82521	305
57	91701	237	22	82216	307
56	91464	239	21	81909	309
55	0.91225	241	0.20	0.81600	311
54	90984	243	19	81289	313
53	90741	245	18	80976	315
52	90496	247	17	80661	317
51	90249	249	16	80344	319
0.50	0.90000	251	0.15	0.8[illegible]	321
49	89749	253	14	79704	323
48	89496	[illegible]	13	79381	325
47	89241	257	12	79056	327
46	88984	[illegible]	11	78729	329
0.45	0.88725	[illegible]	0.10	0.784[illegible]	[illegible]
[illegible]	88464	[illegible]	[illegible]	77736	[illegible]
[illegible]	88[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]936	265	[illegible]	76364	[illegible]
[illegible]	[illegible]	267	[illegible]	75[illegible]96	[illegible]
[illegible]	0.87[illegible]	[illegible]	0.	0.75[illegible]	[illegible]

m in the formula $q = mb\left(H + \tfrac{1}{2}H'\right)\left(H - H'\right)^{\frac{1}{2}}$ see page 118

$\frac{H'}{H}$	0	1	2	3	4	5	6	7	8	9
0.0	[3·330]	·335	·340	·345	·350	·355	·360	·365	·[370]	·366
.1	3·361	·356	350	343	335+	328	320	310	300	290
.2	3·280	·272	264	257	250	243	236	230	223	217
.3	3·210	·204	197+	191	185-	178	172	166	161	156
.4	3·151	·146	141	137	133	129	124	120	117	114
0.5	3·110	·107	·105-	·102	·100	·098	·096	·094	·092	·091
.6	3·090	·089	088	088	087	[087]	087	088	088	089
.7	3·090	·092	094	096	098	101	104	108	112	116
.8	3·120	·126	132	138	144	150	157	164	172	181
9	3·190	·201	212	224	237	250	264	278	294	311
1.0	[3·330]									

See curve opposite page 116

Also see similar table with slightly differing values in W S & I Paper No 200 1907 U.S. [illegible]

the weir should be regarded, and its head h' be applied to H' But it is unnecessary, on account of the limited use of submerged weirs, and the consequent lack of experimental data, to develop this branch of the subject. What has been given above will enable a probable estimate to be made of the discharge in cases where the water accidentally rises above the crest, and further than this the use of submerged weirs cannot be recommended.

Prob. 74. Compute by two methods the discharge over a submerged weir when $b = 8$, $H = 0.46$, and $H' = 0.22$ feet.

ARTICLE 57. ROUNDED AND WIDE CRESTS.

When the inner edge of the crest of a weir is rounded, as at A in Fig. 34, the discharge is materially increased as in the case of orifices (Art. 44), or rather the coefficients of discharge become much larger than those given for the standard sharp crests. The degree of rounding influences so much the amount of increase that no definite values

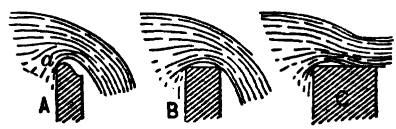


FIG. 34.

can be stated, and the subject is here merely mentioned in order to emphasize the fact that a rounded inner edge is always a source of error. If the radius of the rounded edge is small, the sheet of escaping water leaves it at a point below the top (a in the figure), which has the practical effect of increasing the measured head by a constant quantity. The experiments of FTELEY and STEARNS show that when the radius is less than one-half an inch, the discharge can be computed from the usual weir formula, seven-tenths of the radius being first added to the measured head H.

Two wide-crested weirs with square inner corners are shown in Fig. 34, the one at B being of sufficient width so that the

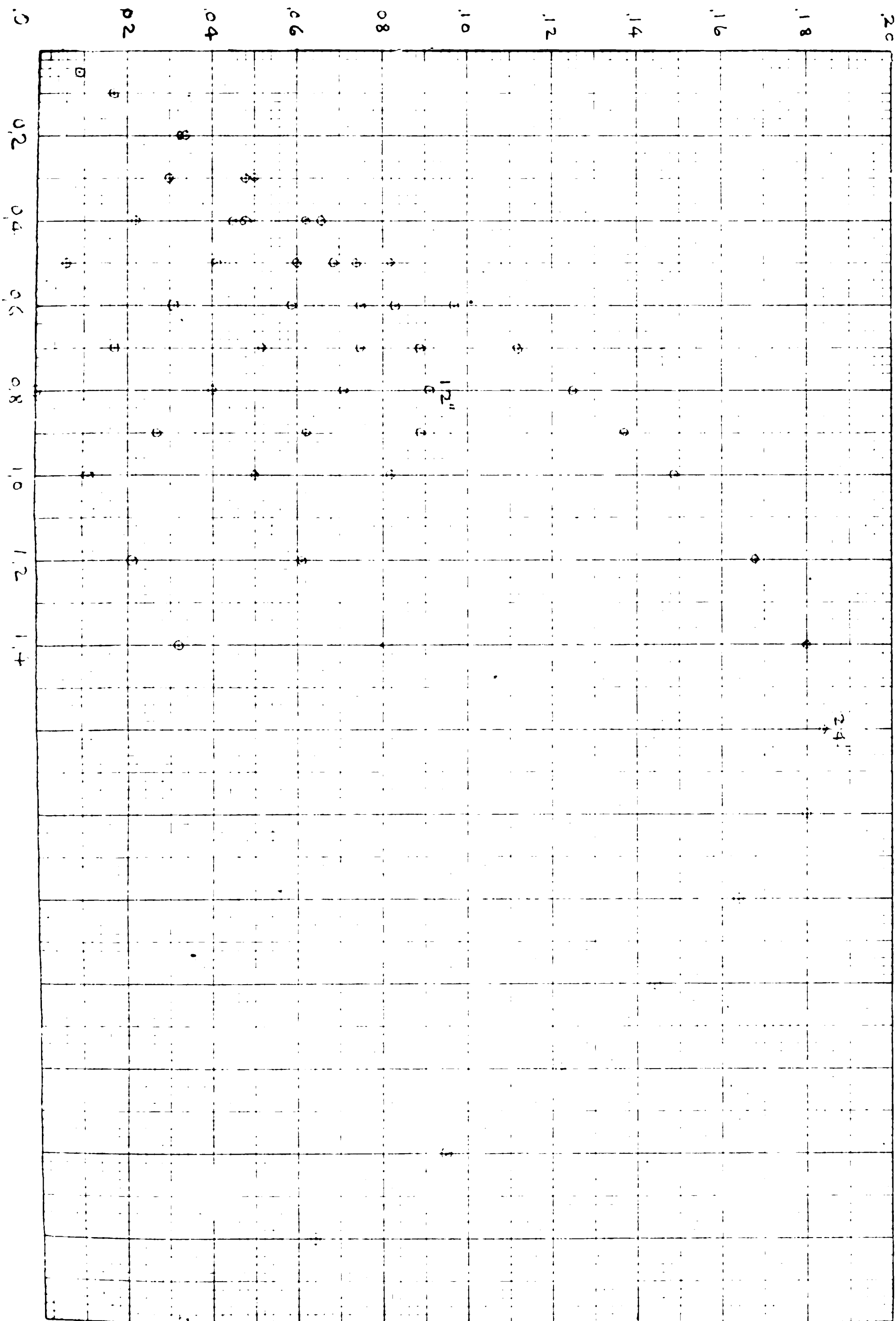
descending sheet may just touch the outer edge, causing the flow to be more or less disturbed, while that at *C* has the sheet adhering to the crest for some distance. In both cases the crest contraction occurs, although water instead of air may fill the space above the inner corner. For *B* the discharge may be equal to or greater than that of the standard weir having the same head *H*, depending upon whether the air has or has not free access beneath the sheet in the space above the crest. For *C* the discharge is always less than that of the standard weir with sharp crest.

The following table is an abstract from the results obtained by FTELEY and STEARNS,* and gives the corrections in feet to be subtracted from the depths on a wide crest, like *C* in Fig. 34, in order to obtain the depths on a standard sharp-crested weir which will discharge an equal volume of water.

TABLE XIII. CORRECTIONS FOR WIDE CRESTS.

Head on wide crest. Feet.	Width of crest in inches.						
	2	4	6	8	10	12	24
0.05	0.010	0.009	0.009	0.009	0.009	0.009	0.009
.10	.016	.018	.017	.017	.017	.017	.017
.20	.012	.029	.031	.032	.033	.033	.034
.30		.030	.041	.045	.047	.048	.050
.40		.022	.045	.055	.060	.062	.066
.50		.006	.041	.060	.069	.074	.082
.60			.031	.059	.075	.083	.097
.70			.017	.052	.075	.089	.112
.80			.000	.040	.071	.091	.125
.90				.027	.062	.089	.137
1.00				.011	.050	.082	.149
1.20					.021	.061	.168
1.40						.032	.180

* Transactions American Society Civil Engineers, 1883, vol. xii. 96.



R. C. CARPENTER, ITHACA N. Y.

These results were obtained by passing a constant volume of water over a standard weir and measuring the head H on the crest; a piece of timber was then brought into place on the lower side of the crest and secured by fastenings, thus forming the wide crest; and the head H being again measured, the increase of depth was thus obtained. This being repeated for different constant volumes the results were plotted and mean curves drawn, from which the table was derived. The weir used was without end contractions, and to such only the conclusions apply with precision. For weirs with end contractions where the air has free access under the sheet at the ends the discharge is probably greater.

Prob. 75. Compute the discharge over a crest 1.5 feet wide for a weir 10 feet long when the head is 0.850 feet, and show that the discharge is about 19 per cent less than that over a standard sharp-crested weir under the same head.

ARTICLE 58. WASTE WEIRS AND DAMS.

Waste weirs are constructed at the sides of canals and reservoirs in order to allow surplus water to escape. They are usually made with wide crests, the inner approach to which is inclined, and the discharge is received upon an apron of timber or masonry. The flow over these wide-crested weirs is always

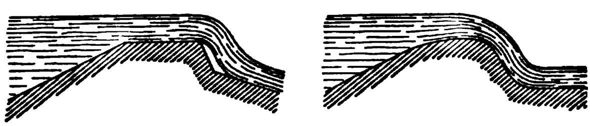


FIG. 35.

much less than for equal depths on standard weirs, and for narrow crests the diminution may be approximately estimated by the use of the table in the preceding article. When the crest is about 3 feet wide, and level, with a rising slope

to its inner edge, and the end contractions are suppressed, the following formula, deduced by FRANCIS, may be applied,

$$q = 3.01 b H^{1.53},$$

in which b and H are to be taken in feet, and q is in cubic feet per second.

In constructing a waste weir the discharge q is generally known or assumed, and it is required to determine b and H. The latter being taken at 1, 2, or 3 feet, as may be judged safe and proper, b is found by

$$b = \frac{q}{3.01 H^{1.53}}.$$

If, for example, q be 87 cubic feet per second, and H be taken as 2 feet, then

$$\log b = \log 87 - \log 3.01 - 1.53 \log 2,$$

from which

$$\log b = 1.0004,$$

whence $b = 10.0$ feet. If, however, H be taken as 1 foot, b is required to be nearly 30 feet.

The ordinary weir formula may be also used for waste-weir calculations with results differing but little from those obtained by the above expression. Or using the approximate general expression from Art. 55,

$$b = \frac{q}{3.33 H^{\frac{3}{2}}}.$$

In this, if q be 87 cubic feet per second, and H be 2 feet, the value of b is found to be 9.24 feet. Evidently no great precision is needed in computing the length of a waste weir, since

it is difficult to determine the exact discharge which is to pass over it, and ample allowance must be made for unusual rains or floods.

When a dam is built across a stream it is often important to arrange its height so that the water level may stand at a certain elevation. In Fig. 36 the line *CC* represents the surface of the stream before the construction of the dam, the depth of water being D, and it is required to find the height of the dam G, so that the surface may be raised the distance d'. If the crest be not submerged, as in the first diagram,

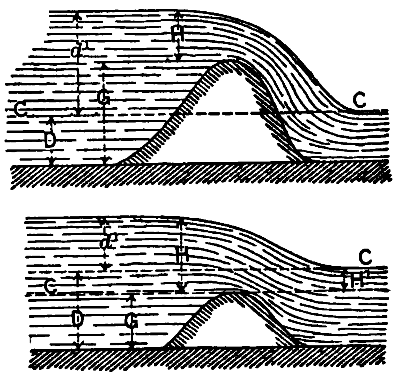


FIG. 36.

$$G = D + d' - H.$$

In this H is to be inserted in terms of the discharge q, or the length b is to be determined as above for an assumed value of H. For the former method,

$$G = D + d' - \left(\frac{q}{3.33b}\right)^{\frac{2}{3}},$$

in which b may be width of the stream or less, as the design requires. If G, D, q, and b be given, this formula may be used to compute d'.

If the height of the dam is small, as in the second diagram of Fig. 36, the crest is submerged, and the last formula will not apply. For this case

$$H = D + d' - G, \qquad H' = D - G;$$

and inserting these heads in the formula (37)′, and solving for G, the following result is found:

$$G = D + \tfrac{2}{3}d' - \frac{2q}{3mb\sqrt{d'}}.$$

In this formula m lies between 3.09 and 3.37, depending on the value of the ratio $H' \div H$, and accordingly a tentative method of solution must be adopted. For example, let $D = 4$ feet, $d' = 1$ foot, $b = 50$ feet, and $q = 400$ cubic feet per second; then, assuming m as 3.33,

$$G = 4 + 0.67 - 1.6 = 3.1 \text{ feet.}$$

Now $H = 4 + 1 - 3.1 = 1.9$ feet, and $H' = 4 - 3.1 = 0.9$, so that the ratio $H' \div H = 0.47$, and hence from Art. 56 the value of m is about 3.12. Using this, the value of G is now computed to be 2.96 feet, which gives $H = 2.04$ feet, and $H' = 1.04$ feet, and $H' \div H = 0.5$, which indicates that no further variation in m will be found. Accordingly 2.96 feet is the required height of the submerged dam.

Prob. 76. If 150 cubic feet per second flow over a waste weir 20 feet long, find the depth of water on the crest.

Prob. 77. A stream 4 feet deep which delivers 150 cubic feet per second is to be dammed so as to raise the water 6 feet higher. Find the height of the dam when the length of the overflow is 12 feet.

ARTICLE 59. THE SURFACE CURVE.

The surface of the water above a weir assumes during the flow a curve whose equation is not known, but some of the laws which govern it may be deduced in the following manner: Let H be the head above the level of the crest measured in

perfectly level water at some distance back of the weir, and let d be the depression or drop of the curve below this level in the plane of the weir (Fig. 37). The discharge per second q can be expressed in terms of H and d by formula (11)′ of Art. 25 by placing H for h_2 and d for h_1. This, multiplied by a coefficient k, gives, if velocity of approach be neglected, the formula

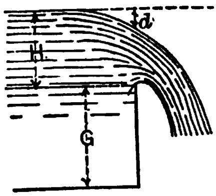


FIG. 37.

$$q = k \cdot \tfrac{2}{3}\sqrt{2g} \cdot b(H^{\frac{3}{2}} - d^{\frac{3}{2}}).$$

This expression, it may be remarked, is the true weir formula, and only the practical difficulties of measuring d prevent its use.

From this formula the value of the drop d in the plane of the weir is found to be

$$d^{\frac{3}{2}} = H^{\frac{3}{2}} - \frac{3q}{2kb\sqrt{2g}}.$$

Let B be the breadth of the feeding canal, G its depth below the crest, and v the mean velocity of approach; then

$$q = B(G + H)v.$$

Inserting this in the equation, replacing $\frac{3}{2k}$ by m, and $\frac{v}{\sqrt{2g}}$ by its value $h^{\frac{1}{2}}$, where h is the velocity-head corresponding to v, the formula becomes

$$d^{\frac{3}{2}} = H^{\frac{3}{2}} - m\frac{B}{b}(G + H)h^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (38)$$

which is an expression for the drop of the curve in terms of the dimensions of the feeding canal and weir, and the heads H and h.

The approximate value of the coefficient m is about 2.2, but precise values of d cannot be computed unless m and H are known with accuracy. The formula, however, serves to exemplify the laws which govern the drop of the curve in the plane of the weir. It shows that the drop increases with the head on the crest and with the length of a contracted weir, that it decreases with the breadth and depth of the feeding canal, and that it decreases with the velocity of approach. It also shows for suppressed weirs, where $B = b$, that the drop is independent of the length of the weir. All of these laws except the last have been previously deduced by the discussion of experiments.

Prob. 78. **Discuss the above formula when $H = 0$; also** when $h = 0$.

ARTICLE 60. TRIANGULAR WEIRS.

Triangular notches are used but little, as in general they are only convenient when the quantity of water to be measured is small. Such a notch when used as a weir must have sharp inner corners, so that the stream may be fully contracted, and the sides should have equal slopes. The angle at the lower vertex should be a right angle, as this is the only case for which coefficients are known with precision. The depth of water above this lower vertex is to be measured by a hook gauge in the usual manner at a point several feet up stream from the notch.

In Art. 23 is deduced a formula for the theoretic discharge through a triangular notch. Making the angle at the vertex a right angle, and applying a coefficient, the theoretic discharge per second is

$$Q = c \cdot \tfrac{8}{15} \sqrt{2g}\, H^{5/2},$$

in which H is the head of water above the vertex. If velocity of approach exists, H may be increased by the velocity-head h as for rectangular weirs.

Experiments made by THOMSON* indicate that the coefficient c varies less with the head than for ordinary weirs; this in fact was anticipated, since the sections of the stream are similar in a triangular notch for all values of H, and hence the influence of the contractions in diminishing the discharge should be nearly constant. As the result of his experiments the mean value of c for heads between 0.2 and 0.8 feet may be taken as 0.592. If, further, 8.02 be put for $\sqrt{2g}$, the discharge in cubic feet per second may be written

$$q = 2.54H^{\frac{5}{2}},$$

in which H must be expressed in feet.

Prob. 79. Find the size of a triangular notch to discharge about 50 cubic feet per second. Also the size of a rectangular weir to discharge the same quantity, when the head is 1.5 feet.

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* British Association Report, 1858, p. 133.

CHAPTER VI.

FLOW THROUGH TUBES.

ARTICLE 61. THE STANDARD SHORT TUBE.

A standard tube is a very short pipe, whose length is about three times its diameter, or of sufficient length so that the escaping jet just fills its outer end, and there issues without contraction. The inner end of the tube is placed flush with the inner side of the reservoir, and is to be a sharp, definite corner, like that of the standard orifice (Art. 34).

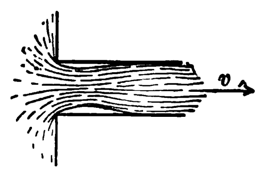

FIG. 38.

The phenomena of flow through such a tube are similar in some respects to those of the flow from the standard orifice, but the discharge is much greater. By observations with glass tubes it is found that the contraction of the jet occurs as in the orifice, although agitation of the water or a shock upon the tube is apt to apparently destroy it, and cause the entire length to be filled. If, however, holes be bored in the tube near its inner end, water does not flow out, but air enters, showing that a negative pressure exists.

Since the issuing jet entirely fills the outer end of the tube, the coefficient of contraction for that section is unity (Art. 35), and hence the coefficient of velocity equals the coefficient of discharge (Art. 37). Numerous experiments by VENTURI, BOSSUT, CASTEL, and others, give the following as a mean value for the standard tube:

$$c = 0.82.$$

This value, however, ranges from 0.83 for low heads and small tubes to 0.80 for high heads and large tubes, its law of variation being probably the same as for orifices (Art. 38), although experiments are wanting from which to state definite values in the form of a table.

A standard orifice gives on the average about 61 per cent of the theoretic discharge, but by the addition of a tube this may be increased to 82 per cent. The effective energy of the jet from the tube is, however, much less than that from the orifice. For, let v be the velocity and h the head, then (Art. 36) for the orifice

$$v = 0.98 \sqrt{2gh}, \text{ whence } \frac{v^2}{2g} = 0.96h;$$

and similarly for the tube,

$$v = 0.82 \sqrt{2gh}, \text{ whence } \frac{v^2}{2g} = 0.67h.$$

Accordingly, the effective energy of the stream from the orifice is 96 per cent of the theoretic energy, while that of the stream from the tube is only 67 per cent. Or if jets be directed vertically upward from a standard orifice and a standard tube, as in Fig. 39, that from the former rises to the height $0.96h$, while that from the latter rises to the height $0.67h$, where h is the head from the level of water AB in the reservoir to the point of exit.

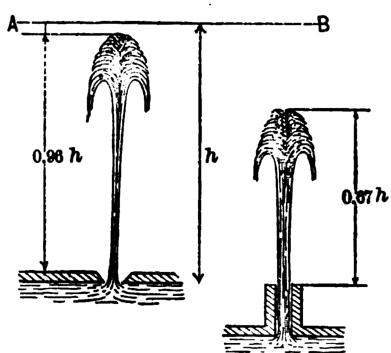

FIG. 39.

The standard tube is not used for the measurement of water, as this can be done with greater precision and convenience by orifices. It is important, however, to know the general laws of flow which have here been set forth, as a starting point in the theory of pipes, and for other purposes. The fact that the tube gives a greater discharge than an orifice is an interesting one, and the reason for this will be explained in Art. 67.

Prob. 80. Compare the effective horse-power of the streams from a standard orifice and tube, the diameter of each being 4 inches and the head 25 feet.

ARTICLE 62. CONICAL CONVERGING TUBES.

Conical converging tubes are used when it is desired to obtain a high efficiency in the energy of the stream of water.

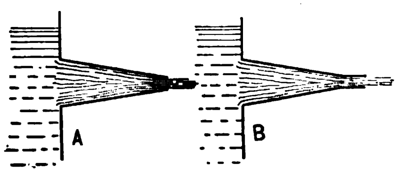

FIG. 40.

At A is shown a simple converging tube, consisting of a frustum of a cone, and at B is a similar frustum, provided with a cylindrical tip. The proportions of these converging tubes, or mouthpieces, vary somewhat in practice, but the cylindrical tip when employed is of a length equal to about 2½ times its inner diameter, while the conical part is eight or ten times the length of that diameter, the angle at the vertex of the cone being between 10 and 20 degrees.

The stream from a conical converging tube like A suffers a contraction at some distance beyond the end. The coefficient of discharge is higher than that of the standard tube, being generally between 0.85 and 0.95, while the coefficient of velocity is higher still. Experiments made by D'AUBUISSON and CASTEL on conical converging tubes 0.04 meters long and 0.0155 meters in diameter at the small end, under a head of 3 meters, give

the following results for the coefficients of discharge and velocity, the former being determined by measuring the actual discharge (Art. 37), and the latter by the range of the jet (Art. 36). The coefficient of contraction, as computed from these, is given in the last column; and this applies to the jet at the smallest section, some distance beyond the end of the tube.

TABLE XIV. COEFFICIENTS FOR CONICAL TUBES.

Angle of Cone.	Discharge c.	Velocity c_1.	Contraction c'.
0° 00′	0.829	0.829	1.00
1 36	0.866	0.867	
4 10	0.912	0.910	
7 52	0.930	0.932	0.998
10 20	0.938	0.951	0.986
13 24	0.946	0.963	0.983
16 36	0.938	0.971	0.966
21 00	0.919	0.972	0.945
29 58	0.895	0.975	0.918
48 50	0.847	0.984	0.861

While these values show that the greatest discharge occurred for an angle of about 13½ degrees, they also indicate that the coefficient of velocity increases with the convergence of the cone, becoming about equal to that of a standard orifice for the last value. Hence the table seems to teach that a conical frustum is not the best form for a mouthpiece to give the greatest velocity.

Under very high heads—over 300 feet—SMITH found the actual discharge to agree closely with the theoretical, or the coefficient of discharge was nearly 1.0, and in some cases slightly greater.* His tubes were about 0.9 feet long, 0.1 feet in

* SMITH's Hydraulics, p. 286.

diameter at the small end and 0.35 feet at the large end, the angle of convergence being 17 degrees. As this implies a contraction of the jet beyond the end, it cannot be supposed that the coefficient of discharge in any case was really as high as his experiments indicate. Under these high heads the cylindrical tip applied to the end of a tube produced no effect on the discharge, the jet passing through without touching its surface.

Prob. 81. If the coefficient of discharge is 0.98 and the coefficient of velocity 0.995, compute the coefficient of contraction.

ARTICLE 63. NOZZLES.

For fire service two forms of nozzles are in use. The smooth nozzle is essentially a conical tube like *A* in Fig. 40, the larger end being attached to a hose, but it is often provided with a cylindrical tip and sometimes the inner end is curved as seen in the upper diagram of Fig. 41. The ring nozzle is a conical tube having an orifice whose diameter is slightly smaller than that of the end of the tube. The experiments of FREEMAN show that the mean coefficient of discharge is about 0.97 for the smooth nozzle and about 0.74 for the ring nozzle.* They also seem to indicate that the simple cone has a higher discharge than any form of curved nozzle.

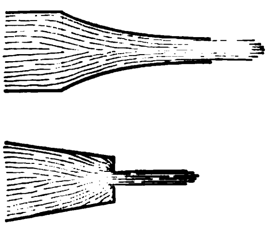

FIG. 41.

The following table contains approximate heights to which jets may be thrown by nozzles according to the investigations of BOX and SHEDD, as quoted in the tables of ELLIS.†

* FREEMAN, Experiments relating to the Hydraulics of Fire Streams. Transactions American Society of Civil Engineers, 1889.

† G. A. ELLIS, Work done by and Power required for Fire Streams, Springfield, Mass., 1878.

This gives the vertical heights in feet reached by jets under different conditions, the first column containing the effective pressure at the entrance to the nozzle, and the second the corresponding effective head.

TABLE XV. VERTICAL HEIGHTS OF JETS FROM NOZZLES.

Pressure in Pounds per Square Inch.	Head in Feet.	From 1-inch Nozzle.		From 1⅛-inch Nozzle.		From 1¼-inch Nozzle.	
		Smooth.	Ring.	Smooth.	Ring.	Smooth.	Ring.
10	23	22	22	22	22	23	22
20	46	43	42	43	43	43	43
30	69	62	61	63	62	63	63
40	92	79	78	81	79	82	80
50	115	94	92	97	94	99	95
60	138	108	104	112	108	115	110
70	161	121	115	125	121	129	123
80	184	131	124	137	131	142	135
90	207	140	132	148	141	154	146
100	230	148	136	157	149	164	155

The effective head at the entrance of a nozzle is the pressure-head plus the velocity-head (Art. 27). Let d be the diameter of the pipe, and d_1 that of the outlet end of the nozzle, and v and v_1 the corresponding velocities. Let h_1 be pressure-head at the entrance; then the effective head is

$$h = h_1 + \frac{v^2}{2g},$$

and the velocity of discharge is

$$v_1 = c_1 \sqrt{2gh} = c_1 \sqrt{2g\left(h_1 + \frac{v^2}{2g}\right)}.$$

Now if h_1 and v are known, v_1 may be computed. But if h_1 is alone known, this equation may be written,

$$\frac{1}{c_1^2}\frac{v_1^2}{2g} = h_1 + \frac{v^2}{2g},$$

also (Art. 19)

$$v_1 d_1^2 = vd^2.$$

Inserting in the first expression the value of v taken from the second, and solving for v_1, gives

$$v_1 = \sqrt{\frac{2gh_1}{\frac{1}{c_1^2} - \frac{d_1^4}{d^4}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Here the last term in the denominator shows the effect of the velocity of approach in the pipe, and if $c_1 = 1$ it agrees with the theoretic expression deduced in Art. 25. In order to use this formula h_1 must be found by observation; one way of doing this is by a pressure gauge at the end of the pipe which reads the pressure p_1 in pounds per square inch; then $h_1 = 2.304p_1$ (Art. 9). If d_1 be small compared with d the formula reduces to $v_1 = c_1\sqrt{2gh_1}$.

The question as to the proper form of curve for a nozzle in order that the velocity may be a maximum is an interesting one. It is thought that this form is similar to that of a jet which rises vertically in a vacuum from an orifice, the sections increasing in size as the velocity diminishes. In the case of the jet the energy of the stream is expended against the constant force of gravity; in the nozzle the constant pressure at the entrance is converted into the energy of the stream. In the jet the velocity is retarded according to a certain law, and hence it seems that in the nozzle the stream should be accelerated according to the same law. The height of the perfect jet is the same as the head h under which it issues from an orifice;

the length of the nozzle l, however, must be short in order to avoid frictional resistances.

Let d_1 be the diameter of the jet at the section where the velocity is $\sqrt{2gh}$. At any distance x above this point let the diameter be y; the velocity in this section is $\sqrt{2g(h-x)}$. Then, since the areas of the sections are inversely as the velocities,

$$\frac{y^2}{d_1^2} = \frac{\sqrt{2gh}}{\sqrt{2g(h-x)}},$$

and from this the value of y is

$$y = d_1 \left(\frac{h}{h-x}\right)^{\frac{1}{4}},$$

which gives the law of variation between the diameters of the different sections. In this formula $y = \infty$ when $x = h$, which should be the case for a theoretically perfect jet rising from an orifice to the level of the reservoir where all its particles are without velocity. Now if L be the length of a nozzle at whose entrance the water has no velocity, and d_1 the diameter at the small end, the diameter at any distance from that end is

$$y = d_1 \left(\frac{L}{L-x}\right)^{\frac{1}{4}} \quad \ldots \ldots \ldots \quad (40)$$

To show the form of profile the following values of y for corresponding values of x are given:

For $x = 0.1L$,	0.3	0.5	0.7	0.8	0.9	0.99
$y = 1.03d_1$,	1.09	1.19	1.35	1.50	1.78	3.16

In practice the nozzle is attached to a hose or pipe whose diameter is d, so that the water enters with a certain velocity. Let l be the length of the nozzle in this case; then in the above equation y equals d when x equals l, and

$$d = d_1 \left(\frac{L}{L-l}\right)^{\frac{1}{4}}.$$

To determine the diameter y at any distance x from the small end, L may be eliminated from the two equations, giving the formula

$$y = d_1 \left(1 - \frac{x}{l} + \frac{d_1^4}{d^4} \frac{x}{l}\right)^{-\frac{1}{4}}, \quad . \quad . \quad . \quad . \quad (40)'$$

from which y may be computed for given values of x, the diameters d_1 and d being first assumed. The value of the length l in practice is often between $6d_1$ and $10d_1$, while d is about $3d_1$. The best relations between l, d_1, and d depend upon frictional resistances, which have here been neglected, and upon considerations of convenience. The following are values of y for corresponding values of x when $d = 3d_1$:

For $x =$	$0.1\,l$	0.3	0.5	0.7	0.8	0.9	1.0
$y =$	$1.03\,d$	1.09	1.19	1.34	1.48	1.74	3.0

These are seen to closely agree with the values deduced above for the nozzle whose diameter d is infinite, and accordingly the equation for that case may be taken as a close expression for the theoretically perfect nozzle whenever d is greater than $3d_1$. The formula (40) has also been deduced by NAGLE from the principle that the velocity in the nozzle should be uniformly accelerated.*

Prob. 82. Compute the coefficient of velocity for a nozzle whose jet rises to a vertical height of 32 feet when the effective pressure at the entrance is 15 pounds per square inch.

Prob. 83. If the coefficient of velocity is 0.98, compute the velocity from a nozzle of 1 inch diameter when attached to a hose of 2½ inches diameter, the pressure at the entrance, as measured by a gauge, being 43.4 pounds per square inch.

Ans. 79.6 feet per second.

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* Transactions American Society of Mechanical Engineers, 1888.

ARTICLE 64. DIVERGING AND COMPOUND TUBES.

In Fig. 42 is shown a diverging conical tube *BC*, and two compound tubes. The compound tube *ABC* consists of two cones, the converging one, *AB*, being much shorter than the diverging one, *BC*, so that the shape roughly approximates to the form of the contracted jet which issues from an orifice in a thin plate. In the tube *AE* the curved converging part *AB* closely imitates the contracted jet, and *BB* is a short cylinder in which all the filaments of the stream are supposed to move in lines parallel to the axis of the tube, the remaining part being a frustum of a cone. The converging part of a compound tube is often called a mouthpiece, and the diverging part an adjutage.

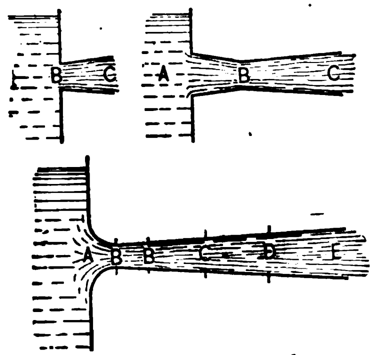


FIG. 42.

Many experiments with these tubes have shown the interesting and phenomenal fact that the discharge and the velocity through the smallest section, *B*, are greater than those due to the head; or, in other words, that the coefficients of discharge and velocity are greater than unity. One of the first to notice this was BERNOULLI in 1738, who found $c = 1.08$ for a diverging tube. VENTURI in 1791 experimented on such tubes, and Venturi
showed that the angle of the diverging part, as also its length, greatly influenced the discharge. He concluded that c would have a maximum value of 1.46 when the length of the diverging part was 9 times its least diameter, the angle at the vertex of the cone being $5° 06'$. EYTELWEIN found $c = 1.18$ for a diverging tube like *BC* in Fig. 42, but when it was used as an

adjutage to a mouthpiece, AB, thus forming a compound tube ABC, he found $c = 1.55$.

The experiments of FRANCIS in 1854 on a compound tube like $ABCDE$ are very interesting.* The curve of the converging part AB was a cycloid, BB was a cylinder, and the diameters at A, B, etc., were

$$A = 1.4 \text{ feet}, \quad C = 0.1454, \quad E = 0.3209$$
$$B = 0.1018, \quad D = 0.2339,$$

The piece BB was 0.1 feet long, and the others each 1 foot; these were made to screw together, so that experiments could be made on different lengths. A sixth piece, EF, not shown in the figure, was also used, which was a prolongation of the diverging cone, its largest diameter being 0.4085 feet. The tubes were of cast-iron, and quite smooth. The flow was measured with the tubes submerged, and the effective head varied from about 0.01 to 1.5 feet. Excluding heads less than 0.1 feet, the following shows the range in value of the coefficients of discharge:

	c for Section BB.	c for Outer End.
For tube AB,	0.80 to 0.94	0.80 to 0.94
For tube AC,	1.43 to 1.59	0.70 to 0.78
For tube AD,	1.98 to 2.16	0.37 to 0.41
For tube AE,	2.08 to 2.43	0.21 to 0.24
For tube AF,	2.05 to 2.42	0.13 to 0.15

The maximum discharge was thus found to occur with the
2.43 tube AE, and to be 2.43 times the theoretic discharge. In general the coefficients increased with the heads, the value 2.08 being for a head of 0.13 feet and 2.43 for a head of 1.36 feet; under 1.39 feet, however, c was found to be 2.26.

The value of g at Lowell, Mass., where these experiments

* Lowell Hydraulic Experiments, 4th Edition, pp. 209–232.

were made, is about 32.162 feet per second. Hence under a 32.162
head of 1.36 feet the theoretic velocity is

$$\sqrt{2gh} = 8.0202 \sqrt{1.36} = 9.36 \text{ feet per second,}$$

while the actual velocity in the section BB was

$$v = 2.43 \times 9.36 = 22.74 \text{ feet per second.}$$

The velocity-head corresponding to this is

$$\frac{v^2}{2g} = (2.43)^2 h = 5.90h.$$

Therefore the flow through the section BB was that due to a head 5.9 times greater than the actual head of 1.36 feet; or, in other words, the energy of the water flowing in BB was 5.9 times the theoretic energy. Here, apparently, is a striking contradiction of the fundamental law of the conservation of energy.

Under high heads the velocity becomes so great that the jet does not touch the sides of the diverging tube, or adjutage, and hence the actual may not exceed the theoretic discharge. It is probable, however, that if the tube be long and its taper very slight an increased discharge can be obtained under a high head.

The explanation of the phenomena of increased velocity and discharge caused by these tubes is simple. It is due to the occurrence of a partial vacuum near the inner end of the adjutage BC. The pressure of the atmosphere on the water in the reservoir thus increases the hydrostatic pressure due to the head, and the increased flow results. The energy at the smallest section is accordingly higher than the theoretic energy, but the excess of this above that due to the head must be expended in overcoming the atmospheric pressure on the outer end of the tube, so that in no case does the available ex-

ceed the theoretic energy. No contradiction of the law of conservation therefore exists.

To render this explanation more definite, let the extreme case be considered where a complete vacuum exists near the inner end of the adjutage, if that were possible, as it perhaps might be with a tube of a certain form. Let h be the head of water in feet on the centre of the smallest section. The mean atmospheric pressure on the water in the reservoir is equivalent to a head of 34 feet (Art. 4). Hence the total head which causes the discharge into the vacuum is $h + 34$ and the velocity of flow is nearly $\sqrt{2g(h + 34)}$. Neglecting the resistances, which are very slight if the entrance be curved, the coefficients of velocity and discharge can now be found; thus:

$$\text{For } h = 100, \quad v = \sqrt{2g \times 134} = 1.16\sqrt{2gh};$$

$$\text{For } h = 10, \quad v = \sqrt{2g \times 44} = 2.10\sqrt{2gh};$$

$$\text{For } h = 1, \quad v = \sqrt{2g \times 35} = 5.92\sqrt{2gh}.$$

The coefficient hence increases as the head decreases. That this is not the case in the above experiments is undoubtedly due to the fact that the vacuum was only partial, and that the degree of rarefaction varied with the velocity. The cause of the vacuum, in fact, is to be attributed to the velocity of the stream, which by friction removes a part of the air from the inner end of the adjutage.

It follows from this explanation that the phenomena of increased discharge from a compound tube could not be produced in the absence of air. The experiment has been tried on a small scale under the receiver of an air-pump, and it was found that the actual flow through the narrow section diminished the more complete the rarefaction. It also follows that it is useless to state any value as representing, even approximately, the coefficient of discharge for such tubes. To secure the highest coefficients, it is thought that the form of the adjutage of

the compound tube should not be conical, but of the shape deduced for the perfect nozzle in Art. 63. The converging part should also properly be of the same form. Then the stream both in contracting and in expanding follows the law of the perfect jet; and hence it may be supposed that the least loss of energy will result, and consequently the greatest flow. This, however, is a mere hypothesis, not yet confirmed by experiment.

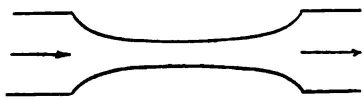


FIG. 43.

Prob. 84. Compute the pressure per square inch in the section BB of FRANCIS' tube when $h = 1.36$ feet and $c = 2.43$. What is the height of the column CD (Fig. 19, Art. 27) that could be lifted by a small pipe inserted at BB?

ARTICLE 65. INWARD PROJECTING TUBES.

Inward projecting tubes, as a rule, give a less discharge than those whose ends are flush with the sides of the reservoir, due to the greater convergence of the lines of direction of the filaments of water. At A and B are shown inward projecting tubes so short that the water merely touches their inner edges, and hence they may more properly be called orifices. Experiment shows that the case at A, where the sides of the tube are normal to the side of the reservoir, gives the minimum coefficient of discharge $c = 0.5$, while for B the value lies between 0.5 and that for the standard orifice at C. The inward projecting cylindrical tube at D has been found to give a discharge of about 72 per cent of the theoretic discharge, while the standard tube (Art. 61) gives 82 per cent. For the

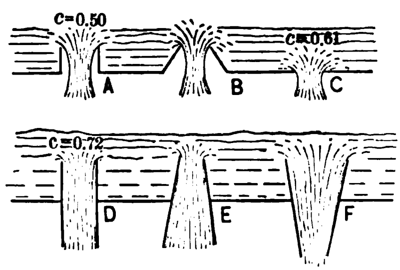


FIG. 44.

tubes E and F the coefficients depend upon the amount of inward projection, and they are much larger than 0.72 for both cases, when computed for the area of the smaller end.

It is usually more convenient to allow a water-main to project inward into the reservoir than to arrange it with its mouth flush to a vertical side. The case D, in Fig. 44, is therefore of practical importance in considering the entrance of water into the main. As the end of such a main has a flange, forming a partial bell-shaped mouth, the value of c is probably higher than 0.72. The usual value taken is 0.82, or the same as for the standard tube (Art. 61). Practically, as will be seen in a later article, it makes little difference which of these is used, as the velocity in such a pipe is slow and the resistance at the mouth is very small compared with the frictional resistances along its length.

Prob. 85. Find the coefficient of discharge for a tube whose diameter is one inch, when the flow under a head of 9 feet is 2.21 cubic feet in 3 minutes and 30 seconds.

ARTICLE 66. EFFECTIVE HEAD AND LOST HEAD.

The terms energy and head are often used as equivalent, although really energy is proportional to head. Thus, if h be the head on an orifice or tube, $\frac{v^2}{2g}$ the velocity head of the issuing jet, and W the weight of water discharged per second, the theoretic energy per second is Wh, the effective or actual energy is $W\frac{v^2}{2g}$, and the lost energy is $W\left(h - \frac{v^2}{2g}\right)$. It is more convenient to deal directly with the heads, omitting the W: thus the effective head in this case is $\frac{v^2}{2g}$, and the lost head is $h - \frac{v^2}{2g}$.

If no losses occur due to friction, contraction, or other causes, the effective head at any point of a tube or pipe is equal to the hydrostatic head h. This effective head may be exerted either in producing pressure or in producing velocity, or part of it in pressure and part in velocity. Thus, as shown in Art. 27,

$$h = h_1 + \frac{v^2}{2g},$$

where h_1 is the pressure-head at the place considered. If there be no motion of the water h equals h_1, and if the flow is so rapid that there be no pressure h equals $\frac{v^2}{2g}$. Owing to the various resistances, however, the effective head $h_1 + \frac{v^2}{2g}$ is generally less than the total head h, and the difference is called the lost head. Thus, at any section of a tube or pipe the head which has been lost is

$$h' = h - \left(h_1 + \frac{v^2}{2g}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

At the end of the tube, or rather outside of the tube, there can be no pressure on the jet, and the loss of head in the flow of the jet hence is

$$h' = h - \frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)'$$

Thus in Art. 46 it was shown that for the standard orifice the loss of energy or head is about 4 per cent, and in Art. 61 it was shown that for the standard tube the loss is about 33 per cent.

In any case the loss of head in a jet from a tube or orifice depends merely on the loss of velocity. Let c_1 be the coefficient of velocity: then for a small orifice or tube

$$v = c_1 \sqrt{2gh},$$

and the effective velocity-head is

$$\frac{v^2}{2g} = c_1^2 h.$$

Consequently the loss of head is

$$h' = h - \frac{v^2}{2g} = (1 - c_1^2)h. \quad . \quad . \quad . \quad . \quad (42)$$

It is sometimes more convenient especially for pipes to express this loss in terms of the velocity-head. The value of h in terms of this is

$$h = \frac{1}{c_1^2}\frac{v^2}{2g},$$

and hence the loss of head is

$$h' = \left(\frac{1}{c_1^2} - 1\right)\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (42)'$$

in which v is the actual velocity of discharge.

For the standard tube (Fig. 38, Art. 61) the coefficient of velocity is equal to the coefficient of discharge whose mean value is 0.82. The effective head of the jet then is

$$\frac{v^2}{2g} = (0.82)^2 h = 0.67h,$$

and the loss of head is

$$h' = (1 - 0.67)h \quad = 0.33h,$$

or

$$h' = \left(\frac{1}{0.67} - 1\right)\frac{v^2}{2g} = 0.49\frac{v^2}{2g}.$$

Hence the loss of head may be said to be either 33 per cent of the total head or 49 per cent of the effective velocity-head; that is, the lost energy is about one-third of the total energy or about one-half of the effective energy.

In reality, work or energy is never lost, but is merely transformed into other forms of energy. In the tube the one-third of the total energy which has been called lost is only lost because it cannot be utilized as work; it is, in fact, transformed into heat, which raises the temperature of the water. And so it is in all cases of lost head: the pressure-head plus the velocity-head is the effective head which can alone be rendered useful; if this be less than the total hydrostatic head, the remainder has disappeared in heat.

Prob. 86. Show that the lost head is nearly equal to the effective head for an inward projecting cylindrical tube.

ARTICLE 67. LOSSES IN THE STANDARD TUBE.

The loss of head in the flow from the short cylindrical tube is large, but not so large as might be expected from theoretical considerations based on the known coefficients for orifices. If the tube has a length of only two diameters the jet does not touch its inner surface, and the flow occurs as from a standard orifice. The velocity in the plane of the inner end is then 61 per cent of the theoretic velocity, since the mean coefficient of discharge is 0.61. Now if the tube be increased in length about one diameter its outer end is filled by the jet, and since the contraction still exists, it might be inferred that the coefficient for that end would be also 0.61: this would give an effective head of $(0.61)^2h$ or $0.37h$, so that the loss of head would be $0.63h$. Actually, however, the coefficient is found to be 0.82 and the loss of head only $0.33h$. It hence appears that further explanation is needed to account for the increased discharge and energy.

It is to be presumed, in the first place, that a loss of about $0.04h$ occurs at the inner end of the tube in the same manner as in the standard orifice, due to retardation of the outer filaments (Art. 46). The effective head at the contracted section

in the tube is then about $0.96h$. If the coefficient of contraction have the value 0.62, as in the orifice, the velocity in that section is greater than at the end of the tube, and, since the velocities are inversely as the areas of the sections, that velocity is

$$v_1 = \frac{0.82}{0.62}\sqrt{2gh} = 1.32\sqrt{2gh},$$

which is nearly one-third larger than the theoretic velocity. The velocity-head at that section then is

$$\frac{v_1^2}{2g} = 1.75h,$$

and consequently the pressure-head is

$$h_1 = 0.96h - 1.75h = -0.79h.$$

There exists therefore a negative pressure or partial vacuum in the tube which is sufficient to lift a column of water to a height of about three-fourths the head. This conclusion has been confirmed by experiment for low heads, and was in fact first discovered experimentally by VENTURI. For high heads it is not valid, since in no event can atmospheric pressure raise a column of water higher than about 34 feet (Art. 4); probably under high heads the coefficient of contraction of the jet in the tube becomes much greater than 0.62.

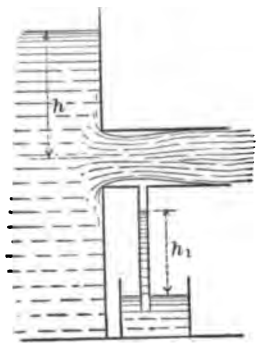


FIG. 45.

The reason of the increased discharge of the tube over the orifice is hence due to the negative pressure or partial vacuum, which causes a portion of the atmospheric head of 34 feet to be added to the head h, so that the flow at the contracted section occurs as if under the head $h + h_1$, as in the diverging

tube (Art. 64). The occurrence of the partial vacuum is attributed to the friction of the sides of the jet on the air. When the flow begins, the jet is surrounded by air of the normal atmospheric pressure which is imprisoned as the jet fills the tube. The friction of the moving water carries some of this air out with it, thus rarefying the remaining air. This rarefaction, or negative pressure, is followed by an increased velocity of flow, and the process continues until the air around the contracted section is so rarefied that no more is removed, and the flow then remains permanent, giving the results ascertained by experiment. The experiments of BUFF have proved that in an almost complete vacuum the discharge of the tube is but little greater than that of the orifice.*

The velocity-head in the contracted section of the jet is thus about $1.75h$, but of this $0.79h$ must be expended in overcoming the atmospheric pressure at the end of the tube, so that the effective head is only $0.96h$. If the retarding influence of the outer end be $0.04h$, or the same as that of the inner end, the effective head is reduced to $0.92h$, while the actual effective velocity-head is $0.67h$. Thus a further loss of $0.25h$ is to be accounted for, and this must be supposed to be due to the enlargement of the section of the jet, and the consequent diminution of velocity, whereby the energy is converted into heat. The partial vacuum causes neither a gain nor loss of head, and the only losses are $0.04h$ at the inner end of the tube, $0.25h$ in the enlargement of the jet, and $0.04h$ at the outer end, or in all $0.33h$. These quantities, of course, are only approximate, as they depend upon the mean coefficients 0.98, 0.62, and 0.82, all of which are liable to variation.

Prob. 87. Discuss the losses of head in an inward projecting tube, taking $c' = 0.6$ and $c = 0.7$.

* See RUHLMANN's Hydromechanik (Hannover, 1879).

ARTICLE 68. LOSS DUE TO ENLARGEMENT OF SECTION.

When a tube or pipe is kept constantly full of water a loss of head is found to result when the section is enlarged so that the velocity is diminished. Let v_1 and v_2 be the velocities in the smaller and larger sections, and h_1 and h_2 the corresponding pressure-heads. The effective head in the first section is the sum of the pressure- and velocity-heads (Arts. 27 and 66), or

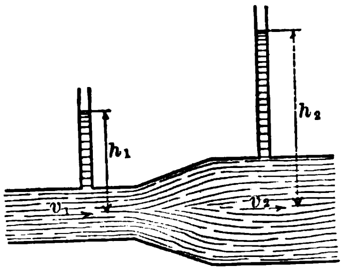


FIG. 46.

$$h_1 + \frac{v_1^2}{2g},$$

and the effective head in the second section is

$$h_2 + \frac{v_2^2}{2g}.$$

If no losses occur, these two expressions are equal; but as the second effective head is always smaller than the first, their difference is the loss of head between the two sections, or the lost head h' is

$$h' = \frac{v_1^2 - v_2^2}{2g} - (h_2 - h_1). \quad . \quad . \quad . \quad . \quad (43)$$

This is a general expression, which gives the loss of head due not only to enlargement, but to all resistances between any two sections of a horizontal tube or pipe. If the difference $h_2 - h_1$ of the pressure columns shown in Fig. 46 is measured, and the velocities determined, the loss of head is thus found in any particular case.

The loss of head due to the sudden enlargement of section,

or rather to the sudden diminution of velocity caused by the enlargement, can be expressed by the formula

$$h' = \frac{(v_1 - v_2)^2}{2g}.$$

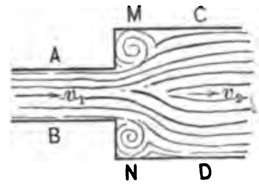


FIG. 47.

To prove this, let p_1 be the unit pressure in AB and p_2 that in CD. At a section MN very near the place of enlargement the unit pressure is also p_1, since the velocity v_1 is maintained for a short distance after leaving AB, its direction, however, being changed so as to form eddies. Let a_2 be the area of the section CD or MN. Then the pressure which acts in the opposite direction to the flow is $a_2(p_2 - p_1)$, and this is the force which causes the velocity to diminish from v_1 to v_2. Now in Art. 32 it was shown that the force which causes W pounds of water to increase in velocity from 0 to v is $\frac{Wv}{g}$, and conversely the same force applied in the opposite direction will cause the velocity to diminish from v to 0. Therefore the value of the pressure $a_2(p_2 - p_1)$ is

$$a_2(p_2 - p_1) = \frac{W}{g}(v_1 - v_2) = \frac{wa_2v_2(v_1 - v_2)}{g},$$

where w is the weight of a cubic unit of water. This expression may be written,

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_2(v_1 - v_2)}{g}$$

or (Art. 9)

$$h_2 - h_1 = \frac{v_2(v_1 - v_2)}{g}.$$

This value of $h_2 - h_1$ inserted in the general equation (43) reduces it to

$$h' = \frac{(v_1 - v_2)^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

which is the formula for loss of head due to sudden enlargement. The loss of energy in this case is similar to that which occurs in the impact of inelastic bodies, work being converted into heat.

Let a_1 and a_2 be the areas of the cross-sections AB and CD. Then $v_1 = \frac{a_2}{a_1} v_2$, and the formula for loss of head in sudden enlargement becomes

$$h' = \left(\frac{a_2}{a_1} - 1\right)^2 \frac{v_2^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)'$$

which is often a more convenient form for practical use. If $a_1 = a_2$ or if $v_2 = 0$ no loss of head results.

If a gradual enlargement of section be made so that no impact occurs, the energy due to the velocity v' is slowly changed into pressure, so that head is not lost. There is, however, no distinct line of division between sudden and gradual enlargement, and for a case like Fig. 46 experiment can alone determine the value of $h_1 - h_2$ and the loss of head. In the last article it was seen that about $0.25h$ is lost in the expansion of the jet between the contracted section and the end of the tube. This seems like a case of gradual enlargement, but as no pressure can exist at the end of the tube the loss of head must be the same as for sudden enlargement of section; in fact $v_1 = 1.32\sqrt{2gh}$ and $v_2 = 0.82\sqrt{2gh}$, whence by the above formula $h' = 0.25h$.

The loss of head due to sudden enlargement may often be very great, as the following example will show. Let the effective head in the section AB be h, all of which exists as velocity, so that $v_1 = \sqrt{2gh}$; let the diameter of AB be 2 inches, and that of CD be 4 inches, so that the area at CD is four times

that at AB, and hence the velocity in CD is $v_2 = \frac{1}{4}\sqrt{2gh}$. The loss of head then is

$$h' = \frac{(v_1 - v_2)^2}{2g} = \tfrac{9}{16}h,$$

so that more than half the energy of the water in AB is lost in shock or impact. At CD the effective head is then $\frac{7}{16}h$, of which $\frac{1}{16}h$ is velocity-head and $\frac{6}{16}h$ is pressure-head. Sudden enlargement of section is therefore to be avoided.

Prob. 88. In a horizontal tube like Fig. 46 the diameters are 6 inches and 12 inches, and the heights of the pressure-columns or piezometers are 12.16 feet and 12.96 feet above the same bench mark. Find the loss of head between the two sections when the discharge is 1.57 cubic feet per second, and also when it is 4.71 cubic feet per second.

ARTICLE 69. LOSS DUE TO CONTRACTION OF SECTION.

When a sudden contraction of section in the direction of the flow occurs, as in Fig. 48, the water suffers a contraction similar to that in the standard tube, and hence in its expansion to fill the smaller section a loss of head results. Let v_1 be the velocity in the larger section and v that in the smaller, while v' is the velocity in the contracted section of the flowing stream; and let a_1, a, and a' be the corresponding areas of the cross-sections. From the formula (44)′ of the last article the loss of head due to the expansion of section from a' to a is

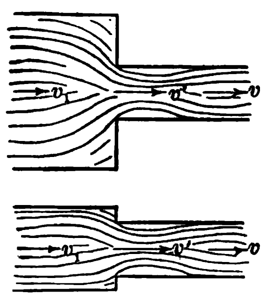


FIG. 48.

$$h' = \left(\frac{a}{a'} - 1\right)^2 \frac{v^2}{2g} = \left(\frac{1}{c'} - 1\right)^2 \frac{v^2}{2g}; \quad . \quad . \quad . \quad (45)$$

in which c' is the coefficient of contraction or the ratio of a' to a.

The value of c' depends upon the ratio between the areas a and a_1. When a is small compared with a_1, the value of c' may be taken at 0.62 as for orifices (Art. 35). When a is equal to a_1 there is no contraction or expansion of the stream, and c' is unity. Let d and d_1 be the diameters corresponding to the areas a and a_1, and let r be the ratio of d to d_1. Then experiments seem to indicate that an expression of the form

$$c' = m + \frac{n}{1.1 - r}$$

gives the law of variation of c' with r. Determining the values of m and n from the two limiting conditions above stated, there is found,

$$c' = 0.582 + \frac{0.0418}{1.1 - r},$$

from which approximate values of c' can be computed. The manner of the variation in the values of c' is indicated by the following tabulation:

For $r =$	0.0,	0.2,	0.4,	0.6,	0.7,	0.8,	0.9,	1.0,
$c' =$	0.62,	0.63,	0.64,	0.67,	0.69,	0.72,	0.79,	1.00.

from which intermediate values may often be taken without the necessity of using the formula.

For a case of gradual contraction of section, such as shown in Fig. 49, the loss of head is less than that given by the above formula, and can only be determined for a given velocity of flow by observing the difference of the heights of the pressure columns. The loss of head then is

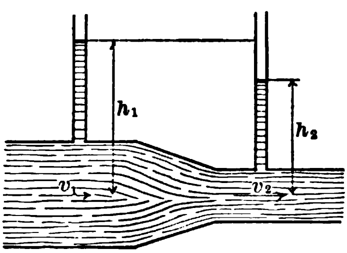


FIG. 49.

$$h' = \frac{v_1^2 - v_2^2}{2g} + h_1 - h_2,$$

as proved in Art. 68. This may be written

$$h' = \left(\frac{a^2}{a_1^2} - 1\right)\frac{v_2^2}{2g} + h_1 - h_2.$$

If the change of section be made so that the stream has no subsequent enlargement, loss of head is avoided, for, as the above discussions show, it is the loss in velocity due to sudden expansion which causes the loss of head.

The loss due to sudden contraction of a tube or pipe is usually much smaller than that due to sudden enlargement. For instance, if the diameter of the larger section be three times that of the smaller, and the velocity in the large section be 2 feet per second, the loss of head when the flow passes from the smaller to the larger section is

$$h' = \frac{(18-2)^2}{2g} = 4.0 \text{ feet.}$$

But if the flow takes place in the opposite direction the coefficient c' is about 0.64, and the loss of head is

$$h' = \left(\frac{1}{0.64} - 1\right)^2 \frac{324}{2g} = 1.6 \text{ feet,}$$

which may be made to vanish by rounding the edges where the change of section occurs.

Prob. 89. Compute the loss of head when a pipe which discharges 1.57 cubic feet per second suddenly diminishes in section from 12 to 6 inches diameter.

ARTICLE 70. PIEZOMETERS.

A piezometer is an instrument for measuring the pressure which exists in a pipe. In its simplest form it consists merely

of a glass tube, as at A, in which the water rises to a height h_1. At B is a form where the tube connecting with the pipe is of metal, which is joined by a flexible hose with a glass tube, which may be placed alongside of a graduated rod to read the height h_1. At C is a common pressure gauge whose dial is graduated so as to read either heights or pressures, as may be desired. When h_1 is found by measurement, the pressure per square unit is computed from the relation $p_1 = wh_1$ (Art. 9). In order to secure accurate results with piezometers, it is necessary that they be inserted into the pipe exactly at right angles; if inclined with or against the current, the height h_1 is greater or less than that due to the actual pressure at the mouth.

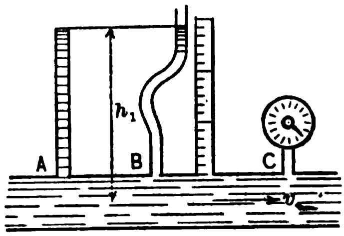


FIG. 49.

If no loss of head occurs between the reservoir and the place where the piezometer is inserted the velocity and discharge through the pipe may be determined. The flow being stopped, the water in the piezometer rises to the height h_1 at the same level as the surface level of the reservoir; when the flow occurs it stands at the height h_2. Then

$$h_1 = h_2 + \frac{v_2}{2g},$$

whence

$$v = \sqrt{2g(h_1 - h_2)}, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and hence the discharge is known for a pipe of given size. It is only in cases of low velocities, however, that this method of gauging the flow is at all applicable, owing to the losses of head which always exist.

The question as to the point from which the pressure-head should be measured deserves consideration. In the figures of the preceding articles h_1 and h_2 have been estimated upward

from the centre of the tube, and it is now to be shown that this is probably correct. Let Fig. 50 represent a cross-section of a tube to which are attached three piezometers as shown. If there be no velocity in the tube or pipe, the water surface stands at the same level in each piezometer, and the mean pressure-head is certainly the distance of that level above the centre of the cross-section. If the water in the pipe be in motion, probably the same would hold true. Referring to formula (43) of Art. 68, and to Fig. 46, it is also seen that if there be no velocity $h' = h_1 - h_2$, which cannot be true unless $h_1 - h_2 = 0$, since there can be no loss of head in the transmission of static pressures; hence h_1 and h_2 cannot be measured from the top of the section. In any event, since the piezometer heights represent the mean pressures, it appears that they should be reckoned upward from the centre of the section. The absolute values of h_1 and h_2 are not generally required, the difference $h_1 - h_2$ being alone used in computations; nevertheless the above considerations are not unimportant.

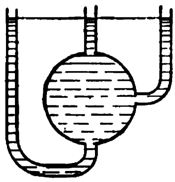


FIG. 50.

The principal application of the piezometer is to the measurement of losses of head, as indicated in Art. 68 for the case of horizontal pipes. The same method applies to inclined pipes, only here the piezometer readings are usually taken above an assumed datum *MN*, as shown in Fig. 51. Let a_1 and a_2 be the areas of any two sections of a pipe, v_1 and v_2 the velocities, H_1 and H_2 the heights of the piezometers above a datum *MN*, and h_1 and h_2 the heights above the axis of the pipes, that is, the mean

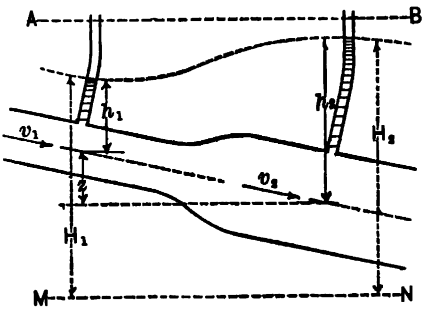


FIG. 51.

pressure-heads. When no flow occurs the piezometers stand in the same level line *AB*. When the flow takes place, delivering W pounds of water per second, the effective energy in the first section is

$$W\left(h_1 + \frac{v_1^2}{2g}\right),$$

and that in the second section is

$$W\left(h_2 + \frac{v_2^2}{2g}\right).$$

Now let z be the vertical distance of the centre of the second section below the first. Were it not for losses the energy in the second section would be

$$W\left(h_1 + \frac{v_1^2}{2g}\right) + Wz.$$

Therefore the energy lost in heat due to friction, enlargement, contraction, and all other causes, between the two sections, is

$$W\left(h_1 + \frac{v_1^2}{2g} + z - h_2 - \frac{v_2^2}{2g}\right);$$

or the loss of head is

$$h' = \frac{v_1^2 - v_2^2}{2g} + h_1 + z - h_2.$$

But from the figure it is seen that

$$h_1 + z - h_2 = H_1 - H_2.$$

Hence the loss of head between the two sections is

$$h' = \frac{v_1^2 - v_2^2}{2g} + H_1 - H_2, \quad . \quad . \quad . \quad . \quad (47)$$

or the same as shown in Art. 68 for horizontal tubes, the piezometer elevations being referred to the same datum.

If the pipe be of the same diameter at the two sections the velocities v_1 and v_2 are equal, and the loss of head is

$$h' = H_1 - H_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)'$$

which is merely the difference of level of the water surfaces

in the piezometers. If the two sections are at the same elevation, or if the second section is lower than the first, this loss is entirely due to resistances which convert the energy into heat. When, however, the second section is higher than the first by the distance z', the head z' is lost in overcoming the force of gravity, and the remainder $h'-z'$ is the portion lost in heat. Piezometers therefore furnish a very convenient method of determining lost head in pipes of uniform section. For pipes of varying section they are rarely applied, as the discharge per second must be measured to find the velocities v_1 and v_2.

In practice it is usually the case that the piezometric tube is simply tapped into the top of the pipe whose flow is to be investigated. It is thought, however, that this may not give the mean pressure throughout the section. In the equations above deduced v_1 and v_2 are the mean velocities in the two sections and h_1 and h_2 the corresponding mean pressure-heads. In order that the piezometer may correctly indicate these mean pressure-heads, they should perhaps be connected with the pipe at the sides and bottom as well as at the top. Piezometric measurements are hence liable to give results more or less uncertain.

If a tube be inserted obliquely to the direction of the current it no longer indicates the true pressure-head, for it is found that the height of the water is greater when the mouth of the tube is inclined toward the current than when inclined away from it. Let θ be the angle between the direction of the flow and the inserted tube. Then the dynamic pressure in the direction of the flow

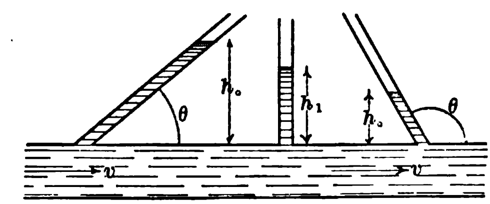


FIG. 52.

is proportional to the velocity-head, and the component of

this in the direction of the tube tends to increase the normal pressure-height h_1 when θ is less than 90° and to decrease it when θ is greater than 90°. Thus

$$h_2 = h_1 + \frac{v^2}{2g} \cos \theta$$

may be written as approximately applicable to the two cases. In this, if the tube be inserted normal to the pipe, $\theta = 90°$ and h_2 becomes h_1, the height due to the static pressure in the pipe; if $v = 0$, the angle θ has no effect upon the piezometer readings. This discussion indicates that when the velocity v is great, piezometric measurements may be affected with errors if the connection be not made truly normal to the direction of the flow.

Prob. 90. In one of the experiments on the compound tube shown in Fig. 53 the areas of the sections a_1 and a_3 were 57.823 square feet, while that of a_2 was 7.047 square feet. When the discharge was 54.02 cubic feet per second the piezometric elevations were:

$$H_1 = 99.838, \qquad H_2 = 98.921, \qquad H_3 = 99.736 \text{ feet.}$$

Show that the head loss was 0.017 feet between a_1 and a_2, and 0.085 feet between a_2 and a_3.

ARTICLE 71. THE VENTURI WATER METER.

It has been shown by HERSCHEL* that a compound tube provided with piezometers may be used for the accurate measurement of water. The apparatus, which is called by him the VENTURI Water Meter, is shown in outline in Fig. 53, and consists of a compound tube (Art. 64) terminated by cylinders, into the top of which are tapped the piezometers H_1 and H_3. Surrounding the small section a_2 is a chamber into which four

* Transactions American Society of Civil Engineers, 1887, vol. xviii. p. 228.

or more holes lead from the top, bottom, and sides of the tube, and from which rises the piezometer H_2. The flow passing through the tube has the velocities v_1, v_2, and v_3 at the sections a_1, a_2, and a_3, and these velocities are inversely as the areas of the sections (Art. 19). When the pressure in a_2 is positive the

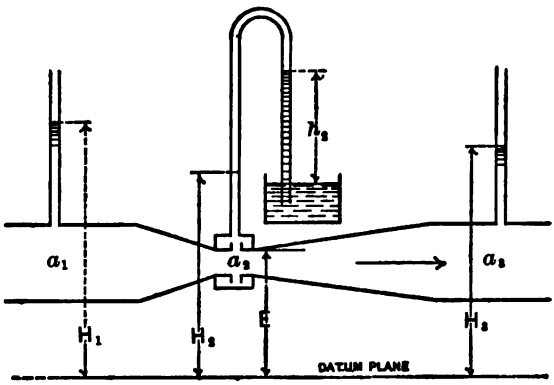


FIG. 53.

water stands in the central piezometer at a height H_2, as shown in the figure; when the pressure is negative the air is rarefied, and a column of water lifted to the height h_2. If E is the height of the top of the section a_2 above the datum the value of H_2 for the case of negative pressure was taken to be $E - h_2$. The apparatus was constructed so that the areas a_1 and a_3 were equal, while a_2 was about one-ninth of these.

To determine the discharge per second through the tube, the areas a_1 and a_2 are to be accurately found by measurements of the diameters; then

$$Q = a_1 v_1, \quad \text{or} \quad Q = a_2 v_2.$$

If no losses of head occur between the sections a_1 and a_2 the quantity h' in the formula of the last article is 0, and

$$0 = \frac{v_1^2 - v_2^2}{2g} + H_1 - H_2.$$

Inserting in this for v_1 and v_2 their values in terms of Q, and then solving for Q, gives the result

$$Q = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g(H_1 - H_2)},$$

which may be called the theoretic discharge. Dividing this expression by a_1 gives the velocity v_1, and dividing it by a_2 gives the velocity v_2. Owing to the losses of head which actually exist, this expression is to be multiplied by a coefficient c; thus:

$$q = c \cdot \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g(H_1 - H_2)} \quad . \quad . \quad . \quad (48)$$

is the formula for the actual discharge per second.

Reference is made to HERSCHEL'S paper, above quoted, for a full description of the method of conducting the experiments. The discharge was actually measured either in a large tank or by a weir; and thus q being known for observed piezometer heights H_1 and H_2, the value of c was computed by dividing the actual by the theoretic discharge. For example, the smaller tube used had the areas

$$a_1 = 0.77288, \qquad a_2 = 0.08634 \text{ square feet};$$

hence the theoretic discharge is

$$Q = 0.086884 \sqrt{2g(H_1 - H_2)},$$

and the coefficient of discharge or velocity is

$$c = \frac{q}{Q}.$$

In experiment No. 1 the value of H_1 was 99.069, while h_2 was 24.509 feet, and the actual discharge was 4.29 cubic feet per second. As E was 84.704, the value of H_2 is 60.195 feet. The theoretic discharge then is

$$Q = 0.086884 \times 8.02 \sqrt{38.874} = 4.345.$$

Dividing 4.29 by this, gives for c the value 0.988. Fifty-five experiments made in this manner, in all of which negative pressure existed in a_2, gave coefficients ranging in value from 0.94 to 1.04, only four being greater than 1.01 and only two less than 0.96.

The larger tube used had the areas $a_1 = 57.823$ and $a_2 = 7.074$ square feet, and the pressure at the central piezometer was both positive and negative. Twenty-eight experiments give coefficients ranging from 0.95 to 0.99, the highest coefficients being for the lowest velocities. In this tube the velocity at the section a_2 ranged from 5 to 34.5 feet per second. The small variation in the coefficients for the large range in velocity indicates that the apparatus may in the future take a high rank as an accurate instrument for the measurement of water. Under low velocities, however, it is not probable that the arrangement of piezometers shown in Fig. 53 will give the best results; in order that H_1 may correctly indicate the mean pressure in a_1, connection seems to be required both at the bottom and sides of the tube like that at a_2. It is thought, moreover, that the elevation E should be measured to the centre of the section rather than to the top. The lower piezometer H_3 is not an essential part of the apparatus and may be omitted, although it was of value in the experiments as showing the total loss of head.

Prob. 91. Given $a_2 = 7.074$ and $a_1 = 57.823$ square feet, $H_2 = 12.204$, $E = 90.909$, and $H_1 = 96.724$ feet, to compute the coefficient of discharge when $q = 243.87$ cubic feet.

CHAPTER VII.

FLOW THROUGH PIPES.

ARTICLE 72. FUNDAMENTAL IDEAS.

The simplest case of flow through a pipe is that where the discharge occurs entirely at the end, there filling the entire section, as in a tube; such pipes are said to be in a condition of full flow. Other cases are those where the discharge is drawn from the pipe at several points along its length, as in the water mains for the supply of towns. Pipes with full flow will be first considered, but most of the principles and tables relating to them apply with but slight modification to water mains. Pipes used in engineering practice are rarely less than $\frac{1}{2}$ inch in interior diameter, and may range from this value upward to 4 feet or more.

The phenomena in a pipe with full flow are apparently simple. The water from the reservoir, as it enters the pipe, suffers more or less contraction, depending upon the manner of connection, as in tubes. Its velocity is then retarded by the resistances of friction and cohesion along the interior surface, so that the discharge at the end is much smaller than in the tube. When the flow becomes permanent the pipe is entirely filled throughout its length; and hence the mean velocity at any section is the same as that at the end, if the size be uniform. This velocity is found to decrease as the

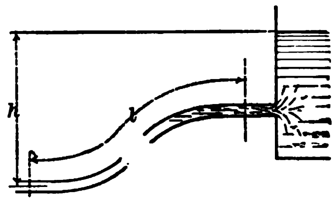


FIG. 54.

length of the pipe increases, other things being equal, and becomes very small for great lengths, which shows that nearly all the head has been lost in overcoming the resistances.

The head which causes the flow is the difference in level from the surface of the water in the reservoir to the centre of the end, when the discharge occurs freely into the air. If h be this head, and W the weight of water discharged per second, the theoretic energy per second is Wh; and if v be the actual velocity of discharge the effective energy is $\frac{Wv^2}{2g}$. The lost energy is then $W\left(h - \frac{v^2}{2g}\right)$, and this has disappeared in heat in overcoming the resistances. In other words, the total head is h, the effective head of the outflowing stream is $\frac{v^2}{2g}$, and the lost head is $h - \frac{v^2}{2g}$. If the lower end of the pipe is submerged, as is often the case, the head h is the difference in elevation between the two water levels.

The length of a pipe is measured along its axis, following all its windings if any. When the length is about two and one-half diameters the pipe is a tube whose coefficient of discharge varies from 0.71 to 0.82, according to the arrangement of its inner end (Art. 65). As the length increases the coefficient of discharge becomes less than from the tube, and for long pipes it becomes very small indeed—indicating that the greater part of the head h is expended in heat in overcoming resistances.

The object of the discussion of flow in pipes is to enable the discharge which will occur under given conditions to be determined, or to ascertain the proper size which a pipe should have in order to deliver a given discharge. The subject cannot, however, be developed with the definiteness which characterizes the flow from orifices and weirs, partly because the

condition of the interior surface of the pipe greatly modifies the discharge, partly because of the lack of experimental data, and partly on account of defective theoretical knowledge regarding the laws of flow. In orifices and weirs errors of two or three per cent may be regarded as large with careful work; in pipes such errors are common, and are generally exceeded in most practical investigations. It fortunately happens, however, that in most cases of the design of systems of pipes errors of five and ten per cent are not important, although they are of course to be avoided if possible, or, if not avoided, they should occur on the side of safety.

Prob. 92. A pipe 500 feet long and 3 inches in diameter discharges about 48 gallons per minute under a head of 4 feet. Compute the coefficient of discharge.

ARTICLE 73. LOSS OF HEAD AT ENTRANCE.

The loss of head which occurs in the upper end of the pipe, due to contraction and resistance of the inner edges, is called the loss at entrance, and this is the same as in a short cylindrical tube under the same velocity of flow. Let c be the coefficient of velocity or discharge for a short tube and v the mean velocity at its outer end, then (Art. 66) the loss of head in the tube is

$$h' = \left(\frac{1}{c^2} - 1\right)\frac{v^2}{2g}.$$

Now this velocity v is the same as that in the pipe into which the tube may be regarded as discharging, and hence this same expression is the loss of head which occurs at the entrance of the pipe, or rather it is the loss at the upper end in a length equal to about three diameters.

The discussions of the last chapter show that the mean value of c is about 0.72 when the tube projects into the reservoir, about 0.82 when the inner end is flush with side of the

reservoir and has square corners, and that it may be nearly 1.00 when the inner end is provided with a bell-shaped mouth. Accordingly the loss of head for a pipe projecting into the reservoir is

$$h' = \left(\frac{1}{0.72^2} - 1\right)\frac{v^2}{2g} = 0.93\,\frac{v^2}{2g};$$

and for a pipe whose end is arranged like a standard tube,

$$h' = \left(\frac{1}{0.82^2} - 1\right)\frac{v^2}{2g} = 0.49\,\frac{v^2}{2g};$$

and for a pipe with a perfect mouthpiece,

$$h' = \left(\frac{1}{1^2} - 1\right)\frac{v^2}{2g} = 0.$$

The loss of head at entrance is hence always less than the velocity-head, and it may be expressed by the formula

$$h' = m\,\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

in which m is 0.93 for the inward projecting pipe, 0.49 for the standard end, and 0 for a perfect mouthpiece. When the condition of the end is not specified the value used for m in the following pages will be 0.5, which supposes that the arrangement is like the standard tube or nearly so. For short pipes, however, it may be necessary to consider the particular condition of the end, and then

$$m = \left(\frac{1}{c^2} - 1\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)'$$

in which c is to be selected from the evidence presented in the last chapter.

It should be noted that the loss of head at entrance is very small for long pipes. For example it is proved by actual gaugings that a pipe 10 000 feet long and 1 foot in diameter discharges about $4\frac{1}{4}$ cubic feet per second under a head of 100 feet. The mean velocity then is

$$v = \frac{4.25}{0.7854} = 5.41 \text{ feet per second,}$$

and the probable loss of head at entrance hence is

$$h' = 0.5 \times 0.01555 \times 5.41^2 = 0.228 \text{ feet},$$

or only one-fourth of one per cent of the total head. In this case the effective velocity-head of the issuing stream is only 0.455 feet, which shows that the total loss of head is 99.545 feet.

Prob. 93. Under a head of 20 feet a pipe 1 inch in diameter and 100 feet long discharges 15 gallons per minute. Compute the loss of head at entrance.

ARTICLE 74. LOSS OF HEAD IN FRICTION.

The loss of head due to the resisting friction of the interior surface of a pipe is usually large, and in long pipes it becomes very great, so that the discharge is but a small percentage of that due to the head. Let h be the total head on a pipe with full flow, $\frac{v^2}{2g}$ the velocity-head of the issuing stream, h' the head lost at entrance, and h'' the head lost in frictional resistances. Then if the pipe be straight and of uniform size, so that no other losses occur,

$$h = \frac{v^2}{2g} + h' + h''.$$

Inserting for h' its value from Art. 73, this equation becomes

$$h = \frac{v^2}{2g} + \left(\frac{1}{c^2} - 1\right)\frac{v^2}{2g} + h'',$$

which is a fundamental formula for the discussion of flow in pipes.

The head lost in friction may be determined for particular cases by measuring the head h, the area a of the cross-section of the pipe, and the discharge per second q. Then q divided by a gives the mean velocity v, and from the above equation

$$h'' = h - \frac{1}{c^2} \cdot \frac{v^2}{2g},$$

which serves to compute h'', the value of c being first selected according to the condition of the end. This method is not applicable to very short pipes because of the uncertainty regarding the coefficient c (Art. 65).

Another method, and the one most generally employed, is by the use of piezometers (Art. 70). A portion of the pipe being selected which is free from sharp curves, two vertical tubes are inserted into which the water rises. The difference of level of the water surfaces in the piezometers is then the head lost in the pipe between them, and this loss is caused by friction alone if the pipe be straight and of uniform size.

By these methods many experiments have been made upon pipes of different sizes and lengths under different velocities of flow, and the discussion of these has enabled the approximate laws to be deduced which govern the loss of head in friction, and tables to be prepared for practical use. These laws are:

1. The loss in friction is proportional to the length of the pipe.
2. It increases nearly as the square of the velocity.
3. It decreases as the diameter of the pipe increases.
4. It increases with the roughness of the interior surface.
5. It is independent of the pressure of the water.

These laws may be expressed by the equation

$$h'' = f\frac{l}{d}\frac{v^2}{2g}, \qquad \ldots \ldots \quad (50)$$

in which l is the length of the pipe, d its diameter, and f is a quantity which depends upon the degree of roughness of the surface. This equation is an empirical one merely; the theoretic expression for h'' is as yet unknown, and it is probable that when discovered it will prove to be of a complex nature.

The values of h'' having been deduced for a number of

cases in the manner just explained, the corresponding values of f can be computed. In this manner it is found that f varies not only with the roughness of the interior surface of the pipe, but also with its diameter, and with the velocity of flow. From the discussions of FANNING, SMITH, and others, the following table of mean values of f has been compiled, which are applicable to clean iron pipes, either smooth or coated with coal-tar varnish, and laid with close joints.

TABLE XVI. FRICTION FACTORS FOR PIPES.

Diameter in Feet.	Velocity in Feet per Second.						
	1.	2.	3.	4.	6.	10.	15.
0.05	0.047	0.041	0.037	0.034	0.031	0.029	0.028
0.1	.038	.032	.030	.028	.026	.024	.023
0.25	.032	.028	.026	.025	.024	.022	.021
0.5	.028	.026	.025	.023	.022	.020	.019
0.75	.026	.025	.024	.022	.021	.019	.018
1.	.025	.024	.023	.022	.020	.018	.017
1.25	.024	.023	.022	.021	.019	.017	.016
1.5	.023	.022	.021	.020	.018	.016	.015
1.75	.022	.021	.020	.018	.017	.015	.014
2.	.021	.020	.019	.017	.016	.014	.013
2.5	.020	.019	.018	.016	.015	.013	.012
3.	.019	.018	.016	.015	.014	.013	.012
3.5	.018	.017	.016	.014	.013	.012	
4.	.017	.016	.015	.013	.012	.011	
5.	.016	.015	.014	.013	.012		
6.	.015	.014	.013	.012	.011		

The quantity f may be called the friction factor, and the table shows that its value ranges from 0.05 to 0.01 for new clean pipes. A rough mean value, often used in approximate computations, is

$$\text{Friction factor } f = 0.02.$$

It is seen that the tabular values of f decrease both when the diameter and when the velocity increases, and that they vary most rapidly for small pipes and low velocities. The probable error of a tabular value of f is liable to be about one unit in the third decimal place, which is equivalent to an uncertainty of ten per cent when $f = 0.011$, and to five per cent when $f = 0.021$. The effect of this is to render computed values of h'' liable to the same uncertainties; but the effect upon computed velocities and discharges is much less, as will be seen in Art. 76.

To determine, therefore, the probable loss of head in friction, the velocity v must be known, and f is taken from the table for the given diameter of pipes. The formula

$$h'' = f\frac{l}{d} \cdot \frac{v^2}{2g}$$

then gives the probable loss of head in friction. For example, let $l = 10\,000$ feet, $d = 1$ foot, $v = 5.41$ feet. Then, from the table, f is 0.021, and

$$h'' = 0.021 \times \frac{10\,000}{1} \times 0.455 = 96 \text{ feet},$$

which is to be regarded as an approximate value, liable to an uncertainty of five per cent.

The theory of the internal frictional resistances, as far as understood, indicates that the energy which is thus transformed into heat is expended in two ways: first, in the direct friction along the interior surface; and second, in impact caused by an unsteady motion of the particles of water. Under very low velocities the motion is in lines parallel to the axis of the pipe, so that resistance is met only along the surface, but under ordinary conditions the motion of many of the particles is sinuous, whereby internal friction or impact is also produced. Experiments devised by REYNOLDS enable this sinuous motion to be actually seen, so that its existence is beyond question.

Prob. 94. Determine the actual loss of head in friction from the following experiment: $l = 60$ feet, $h = 8.33$ feet, $d = 0.0878$ feet, $q = 0.03224$ cubic feet per second, and $c = 0.8$. Compute by help of the table the probable loss for the same data.

ARTICLE 75. OTHER LOSSES OF HEAD.

Thus far the pipe has been supposed to be straight and of uniform size, so that no losses of head occur except at entrance and in friction. But if the pipe vary in diameter, or have sharp curves, or contain valves, further losses occur, which are now to be considered.

Sudden enlargements and contractions of section cause losses of head which may be ascertained by the rules of Arts. 68 and 69. These are of infrequent occurrence in pipes, the usual method of passing from one size to another being by means of a "reducer," which is a conical frustum several feet long, whereby the velocity is slowly changed without expending energy in impact.

The loss of head caused by easy curves is very slight, and need not be taken into account. For sharp curves the loss is small, rarely exceeding twice the velocity-head for a single curve, but when many such curves occur the item of loss thus caused may be important. According to the investigations of WEISBACH, the loss of head due to a curve of one-fourth of a circle may be written

$$h''' = n\frac{v^2}{2g},$$

in which n is a number whose value is given below for different values of $\frac{d}{2R}$, where R is the radius of the curve of the centre line of the pipe, and d is its diameter:

For $\frac{\frac{1}{2}d}{R} = 0,\ 0.1,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,$

$n = 0,\ 0.13,\ 0.16,\ 0.21,\ 0.29,\ 0.44,\ 0.66,\ 0.98,\ 1.41,\ 1.98$

These coefficients, however, were derived for small pipes, and it is probable that for large pipes the loss of head may be less than they indicate.

In Fig. 55 are shown three kinds of valves for regulating the flow in pipes: at A a valve consisting of a vertical sliding-gate, at B a cock-valve formed by two rotating segments, and

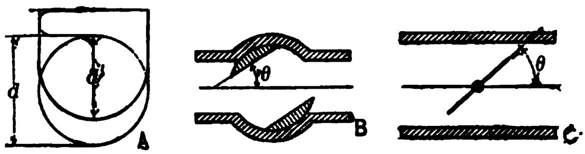


FIG. 55.

at C a throttle-valve or circular disk which moves like a damper in a stove-pipe. The loss of head due to these may be very large when they are sufficiently closed so as to cause a sudden change in velocity. It may be expressed by

$$h''' = n\frac{v^2}{2g},$$

in which n has the following values, as determined by the experiments of WEISBACH.* For the sluice-valve let d' be the vertical distance that the gate is lowered below the top of the pipe; then

For $\frac{d'}{d} =$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
$n =$	0.0	0.07	0.26	0.81	2.1	5.5	17	98

For the cock-valve let θ be the angle through which it is turned, as shown in the figure; then

For $\theta =$	0°	10°	20°	30°	40°	50°	55°	60°	65°
$n =$	0	0.29	1.6	5.5	17	53	106	206	486

* Mechanics of Engineering, vol. i., COXE's translation, p. 902.

In like manner, for the throttle-valve the coefficients are:

For $\theta =$	5°	10°	20°	30°	40°	50°	60°	65°	70°
$n =$	0.24	0.52	1.5	3.9	11	33	118	256	750

The number n hence rapidly increases and becomes infinity when the valve is fully closed, but as the velocity is then zero there is no loss of head. The velocity v here, as in other cases, refers to that in the main part of the pipe, and not to that in the contracted section formed by the valve.

An accidental obstruction in a pipe may be regarded as causing a sudden change of section, and the loss of head due to it is, by Art. 68,

$$h''' = \left(\frac{a}{a'} - 1\right)^2 \frac{v^2}{2g} = n\frac{v^2}{2g},$$

where a is the area of the section of the pipe, and a' that of the diminished section. This formula shows that when a' is one-half of a, the loss of head is equal to the velocity-head, and that n rapidly increases as a' diminishes.

In the following pages the symbol h''' will be used to denote the sum of all the losses of head due to curvature, valves, and contractions of section. Then

$$h''' = n\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51$$

in which n will denote the sum of all the coefficients due to these causes. In case no mention is made regarding these sources of loss they are supposed not to exist, so that both n and h''' are simply zero.

Prob. 95. Compute for the data of the last problem the loss of head caused by a curve whose radius is 2 feet.

Ans. 0.013 feet.

ARTICLE 76. FORMULA FOR VELOCITY.

The mean velocity in a pipe can now be deduced for the condition of full flow. The total head being h, and the effective velocity-head of the issuing stream being $\frac{v^2}{2g}$, the lost head is $h - \frac{v^2}{2g}$, and this must be equal to the sum of its parts, or

$$h - \frac{v^2}{2g} = h' + h'' + h''' . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Substituting in this the values of h', h'', and h''' from the preceding articles, it becomes

$$h - \frac{v^2}{2g} = m\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g} + n\frac{v^2}{2g}; \quad . \quad . \quad (52)'$$

and by solving for v there is found

$$v = \sqrt{\frac{2gh}{1 + m + f\frac{l}{d} + n}}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

which is a general formula for the velocity of flow.

In this formula n will be taken as 0, unless otherwise stated; that is, no losses of head occur except at entrance and in friction. The formula for pipes which are essentially straight and of uniform size throughout then is

$$v = \sqrt{\frac{2gh}{1.5 + f\frac{l}{d}}}. \quad . \quad . \quad . \quad . \quad . \quad (53)'$$

Here m is taken as 0.5, which is to be regarded as its mean value in accordance with the discussion in Art. 73.

In this formula the friction factor f is a function of v to be taken from the table in Art. 74, and hence v cannot be directly

computed, but must be obtained by successive approximations. For example, let it be required to compute the velocity of discharge from a pipe 3000 feet long and 6 inches in diameter under a head of 9 feet. Here $l = 3000$, $d = 0.5$, and $h = 9$; taking for f the rough mean value 0.02, the formula gives

$$v = \sqrt{\frac{2 \times 32.16 \times 9}{1.5 + 0.02 \times 3000 \times 2}} = 2.2.$$

The approximate velocity is hence 2.2 feet per second, and entering the table with this, the value of f is found to be 0.026. Then the formula gives

$$v = \sqrt{\frac{2 \times 32.16 \times 9}{1.5 + 0.026 \times 3000 \times 2}} = 1.92.$$

This is to be regarded as the probable value of the velocity, since the table gives $f = 0.026$ for $v = 1.92$. In this manner by one or two trials the value of v can be computed so as to agree with the corresponding value of f.

The error in the computed velocity due to an error of one unit in the last decimal of the factor f is always relatively less than the error in f itself. For instance, if v be computed for the above example with $f = 0.025$, its value is found to be 1.96 feet per second, or two per cent greater than 1.92. In general, the percentage of error in v is less than one-half of that in f. It hence appears that computed velocities are liable to probable errors ranging from one to five per cent, owing to imperfections in the tabular values of f, for new clean pipes. This uncertainty is as a rule still further increased by various causes, so that five per cent is to be regarded as a common probable error in computations of velocity and discharge from pipes.

Velocities greater than 15 feet per second are very unusual in pipes, and but little is known as to the values of f for such cases. For velocities less than one foot per second, the values

of f are also not understood, so that little reliance can be placed upon computations. The usual velocity in water mains is less than five feet per second, it being found inadvisable to allow swifter flow on account of the great loss of head in friction.

To illustrate the use of the general formula, let the pipe in the above example be supposed to have a curve of 6 inches radius, and to contain a gate valve which is half closed. Then from Art. 75, $n = 0.29$ for the curve and $n = 2.1$ for the valve, or in the formula n is to be put as 2.39. The velocity is now found to be

$$v = \sqrt{\frac{2 \times 32.16 \times 9}{3.89 + 0.026 \times 6000}} = 1.90 \text{ feet per second};$$

which is but a trifle less than that found before. The closing of the sluice gate to one-half its depth hence but slightly influences the velocity, while the effect of the curve is scarcely perceptible. With a shorter pipe, however, the influence of these would be more marked.

Prob. 96. Compute the velocity for the data of the last example if the pipe be 1000 feet long.

Prob. 97. Compute the velocity for a pipe 15 000 feet long and 18 inches in diameter under a head of 230 feet.

Ans. 9.57 feet per second.

ARTICLE 77. COMPUTATION OF DISCHARGE.

The discharge per second from a pipe of given diameter is found by multiplying the velocity of discharge by the area of the cross-section of the pipe, or

$$q = \tfrac{1}{4}\pi d^2 v = 0.7854 d^2 v, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

in which v is to be found by the method of the last article.

For example, let it be required to find the discharge in gallons per minute from a clean pipe 3 inches in diameter and

500 feet long under a head of 4 feet. Here $d = 0.25$, $l = 500$, and $h = 4$. Then for $f = 0.02$, the velocity is found to be 2.5 feet per second; again taking from the table $f = 0.027$, the velocity is 2.15 feet per second. The discharge in cubic feet per second is

$$q = 0.7854 \times 0.25^2 \times 2.15 = 0.106;$$

and in gallons per minute,

$$q = 0.106 \times 7.48 \times 60 = 47.6.$$

This is the probable result, which is liable to the same uncertainty as the velocity—say about three per cent; so that strictly the discharge should be written 47.6 ± 1.4 gallons.

By inserting the value of v in the above expression for q it becomes

$$q = \tfrac{1}{4}\pi d^2 \sqrt{\frac{2gh}{1 + m + f\dfrac{l}{d} + n}};$$

and from this the value of the head required to produce a given discharge is

$$h = \frac{16}{2g\pi^2}\left(1 + m + f\frac{l}{d} + n\right)\frac{q^2}{d^4}.$$

These formulas are not more convenient for practical computations than the separate expressions for v, q, and h previously established, since in any event v must be computed in order to select f from the table. They serve, however, to exhibit the general laws which govern the discharge.

Prob. 98. Compute the probable discharge from a pipe 1 inch in diameter and 1000 feet long under a head of 40 feet.

Prob. 99. What head is required to discharge 3 gallons per minute through a pipe 1 inch in diameter and 1000 feet long?

Ans. 11.3 feet.

ARTICLE 78. COMPUTATION OF DIAMETER.

It is an important practical problem to determine the diameter of a pipe to discharge a given quantity of water under a given head and length. The last equation above serves to solve this case, as all the quantities in it except d are known. This may be written in the form

$$d^5 = \Big[(1 + m + n)d + fl\Big]\frac{16q^2}{2g\pi^2 h};$$

or placing for m and $2g$ their mean values and neglecting n, it becomes

$$d = 0.479\Big[(1.5d + fl)\frac{q^2}{h}\Big]^{\frac{1}{5}}, \quad . \quad . \quad . \quad . \quad (55)$$

which is the formula for computing d when h, l, and d are in feet and q is in cubic feet per second. The value of the friction factor f may be taken as 0.02 in the first instance, and the d in the right-hand member being neglected, an approximate value of the diameter is computed. The velocity is next found by the formula

$$v = \frac{q}{\frac{1}{4}\pi d^2},$$

and from Table XVI. the value of f thereto corresponding is selected. The computation for d is then repeated, placing in the right-hand member the approximate value of d. Thus by one or two trials the diameter is computed which will satisfy the given conditions.

For example, let it be required to determine the diameter of a pipe which, under the condition of full flow, will deliver 500 gallons per second, its length being 4500 feet and the head 24 feet. Here the value of q is

$$q = \frac{500}{7.481} = 66.84 \text{ cubic feet.}$$

The approximate value of d then is

$$d = 0.479\left(\frac{0.02 \times 4500 \times 66.84^2}{24}\right)^{\frac{1}{5}} = 3.35 \text{ feet.}$$

From this the velocity of flow is

$$v = \frac{66.84}{0.7854 \times 3.35^2} = 7.6 \text{ feet per second,}$$

and from the table the value of f for this diameter and velocity is found to be 0.013. Then

$$d = 0.479\left[(1.5 \times 3.35 + 0.013 \times 4500)\frac{66.84^2}{24}\right]^{\frac{1}{5}},$$

from which $d = 3.125$ feet. With this value of d the velocity is now found to be 8.71 feet, so that no change results in the value of f. The required diameter of the pipe is therefore 3.1 feet, or about 37 inches; but as the regular market sizes of pipes furnish only 36 inches and 40 inches, one of these must be used, and it will be on the side of safety to select the larger.

It will be well in determining the size of a pipe to also consider that the interior surface may become rough by erosion and incrustation, thus increasing the value of the friction factor and diminishing the discharge. The increase in f from these causes is not likely to be so great in a large pipe as in a small one, but it is thought that for the above example they might be sufficient to make f as large as 0.03. Applying this value to the computation of the diameter from the given data there is found $d = 3.6$ feet $=$ about 43 inches.

The sizes of pipes generally found in the market are $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, 3, 4, 6, 8, 10, 12, 16, 18, 20, 24, 27, 30, 36, 40, 44, and 48 inches, while intermediate or larger sizes must be made to order. The computation of the diameter is merely a guide to enable one of these sizes to be selected, and therefore it

is entirely unnecessary that the numerical work should be carried to a high degree of precision. In fact, three-figure logarithms are usually sufficient to determine reliable values of d.

Prob. 100. Compute the diameter of a pipe to deliver 50 gallons per minute under a head of 4 feet when its length is 500 feet. Also when its length is 5000 feet.

ARTICLE 79. SHORT PIPES.

A pipe is said to be short when its length is less than about 500 times its diameter, and very short when the length is less than about 50 diameters. In both cases the coefficient c should be estimated according to the condition of the upper end as precisely as possible, and the length l should not include the first three diameters of the pipe, as that portion properly belongs to the tube which is regarded as discharging into the pipe. In attempting to compute the discharge for such pipes, it is often found that the velocity is greater than given in Table XVI., and hence that the friction factor f cannot be determined. For this reason no accurate estimate can be made of the discharge from short pipes under high heads, and fortunately it is not often necessary to use them in engineering constructions.

For example, let it be required to compute the velocity of flow from a pipe 1 foot in diameter and 100 feet long under a head of 100 feet, the upper end being so arranged that $c = 0.80$, and hence $m = 0.56$ (Art. 73). Neglecting n, since the pipe has no curves or valves, the formula for the velocity becomes

$$v = \sqrt{\frac{2gh}{1.56 + f\frac{l}{d}}};$$

and using for f the rough mean value 0.02,

$$v = \sqrt{\frac{64.32 \times 100}{1.56 + 0.02 \times 97}} = 42.9 \text{ feet per second.}$$

Now there is absolutely no experimental knowledge regarding the value of the friction factor f for such high velocities, but judging from the table it is probable that f may be about 0.015. Using this instead of 0.02 gives for v the value 46 feet per second. The uncertainty of this result should be regarded as at least ten per cent.

For very short pipes there are on record a number of experiments by EYTELWEIN and others, from which the coefficients of discharge have been deduced. The upper end of the pipe being in all cases arranged like the standard tube, these experiments give the following as mean values of the velocity:

For $l = 3d,\qquad v = 0.82\sqrt{2gh}$
For $l = 12d,\qquad v = 0.77\sqrt{2gh}$
For $l = 24d,\qquad v = 0.73\sqrt{2gh}$
For $l = 36d,\qquad v = 0.68\sqrt{2gh}$
For $l = 48d,\qquad v = 0.63\sqrt{2gh}$
For $l = 60d,\qquad v = 0.60\sqrt{2gh}$

These coefficients were deduced for small pipes under low heads, and are to be regarded as liable to a variation of several per cent; for large pipes and high heads they are all probably too large.

The general equation for the velocity of discharge deduced in Art. 76 may be applied to very short pipes by writing $l - 3d$ in place of l, and placing for m its value in terms of c. It then becomes

$$v = \sqrt{\frac{2gh}{\frac{1}{c^2} + f\frac{l-3d}{d}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

If in this l equals $3d$, the velocity is

$$v = c\sqrt{2gh},$$

which is the same as for a short tube. If $l = 12d$, $f = 0.02$, and $c = 0.82$, it gives $v = 0.774\sqrt{2gh}$, which agrees well with the mean value above stated.

Prob. 101. Compute the discharge per second for a pipe 1 inch in diameter and 40 inches long under a head of 4 feet.

ARTICLE 80. LONG PIPES.

For long pipes the loss of head at entrance becomes very small compared with that lost in friction, and the velocity-head is also small. The formula for velocity deduced in Art. 76 is

$$v = \sqrt{\frac{2gh}{1.5 + f\frac{l}{d}}},$$

in which the first term in the denominator represents the effect of the velocity-head and the entrance-head, the mean value of the latter being 0.5. Now it may safely be assumed that 1.5 may be neglected in comparison with the other term, when the error thus produced in v is less than one per cent. Taking for f its mean value, this will be the case when

$$\frac{\sqrt{1.5 + 0.02\frac{l}{d}}}{\sqrt{0.02\frac{l}{d}}} = 1.01, \quad \text{from which} \quad \frac{l}{d} = 3750.$$

Therefore when l is greater than about $4000d$ the pipe will be called long.

For long pipes the velocity under full flow hence is given by the formula

$$v = \sqrt{\frac{2gdh}{fl}} = 8.02\sqrt{\frac{dh}{fl}}, \quad . \quad . \quad . \quad . \quad (57)$$

and the discharge per second is,

$$q = \tfrac{1}{4}\pi d^2 v = 6.30\sqrt{\frac{d^5h}{fl}}. \quad . \quad . \quad . \quad . \quad (57)'$$

For computing the diameter required to deliver a given discharge the formula is

$$d = 0.479 \left(\frac{flq^2}{h}\right)^{\frac{1}{5}}. \quad . \quad . \quad . \quad . \quad . \quad (57)''$$

These equations show that for very long pipes the discharge varies directly as the $2\frac{1}{2}$ power of the diameter, and inversely as the square root of the length.

In the above formulas d, h, and l are to be taken in feet, q in cubic feet per second, and f is to be found from the table in Art. 74, an approximate value of v being first obtained by taking f as 0.02. It should not be forgotten that these expressions are of an empirical nature, and do not necessarily represent the true laws of flow; but at present they seem to be the representation of these laws which for long pipes best agrees with experiments. The value of h in these formulas is also really the friction-head h'', since in their deduction the other heads, h', h''', and $\frac{v^2}{2g}$, have been neglected; these, however, although often very small, can never be really zero.

Prob. 102. Compute the probable discharge from a pipe 26 500 feet long and 18 inches in diameter under a head of 324.7 feet. Ans. 14.7 cubic feet per second.

Prob. 103. Compute the diameter required to deliver 15 000 cubic feet per hour through a pipe 26 500 feet long under a head of 324.7 feet. If this quantity is carried in two pipes of equal diameter, what should be their size?

ARTICLE 81. RELATIVE DISCHARGING CAPACITIES.

For orifices and short tubes the discharge under a given head varies as the square of the diameter. In pipes of equal length under given heads the discharges vary more rapidly

than the squares of the diameters, owing to the influence of friction. For a long pipe the formula for discharge is

$$q = \tfrac{1}{4}\pi \sqrt{\frac{2ghd^5}{fl}},$$

which shows that if f be constant the discharge varies as the $2\frac{1}{2}$ power of the diameter. This is a useful approximate rule for comparing the relative discharging capacities of pipes.

Thus if there be two pipes with diameters d_1 and d_2 the rule gives

$$q_1 : q_2 = d_1^{\frac{5}{2}} : d_2^{\frac{5}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and from this

$$q_2 = q_1 \left(\frac{d_2}{d_1}\right)^{\frac{5}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)'$$

For example, if there be two pipes of the same length under the same head, the first one foot and the second two feet in diameter,

$$q_2 = q_1 \left(\frac{2}{1}\right)^{\frac{5}{2}} = 5.7q_1,$$

or the second pipe discharges nearly six times as much as the first. In other words, six pipes of 1 foot diameter are about equivalent to one pipe of 2 feet diameter.

As the friction factor f is not constant, the above rule is not exact; for, as the formula shows,

$$q_1 : q_2 = \left(\frac{d_1^5}{f_1}\right)^{\frac{1}{2}} : \left(\frac{d_2^5}{f_2}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

from which

$$q_2 = q_1 \left(\frac{d_2}{d_1}\right)^{\frac{5}{2}} \left(\frac{f_1}{f_2}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)'$$

Now as the values of f vary not only with the diameter but with the velocity, a solution, cannot be made except in particular cases. For the above example let the velocity be about

3 feet per second; then from the table $f_1 = 0.023$ and $f_2 = 0.019$, and

$$q_2 = q_1\,(2)^{\frac{5}{2}}\,(1.2)^{\frac{1}{2}} = 6.2q_1,$$

or the two-foot pipe discharges more than six times as much as the one-foot pipe.

Prob. 104. How many pipes, 6 inches in diameter, are equivalent in discharging capacity to one pipe 24 inches in diameter?

ARTICLE 82. A COMPOUND PIPE.

A compound pipe is one laid with different sizes in different portions of its length. In such the change from one size to another is to be made gradually by a reducer, so that losses of head due to sudden enlargement or contraction are avoided (Art. 68). Let d_1, d_2, d_3, etc., be the diameters; l_1, l_2, l_3, etc., the corresponding lengths, the total length being $l_1 + l_2 +$ etc. Let v_1, v_2, etc., be the velocities in the different sections. Neglecting the loss of head at entrance, the total head h may be placed equal to the loss of head in friction, or

$$h = f_1 \frac{l_1}{d_1} \frac{v_1^2}{2g} + f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} + \text{etc.} \quad . \quad . \quad . \quad (60)$$

Now if the discharge per second be q,

$$v_1 = \frac{4q}{\pi d_1^2}, \qquad v_2 = \frac{4q}{\pi d_2^2}, \quad \text{etc.}$$

Substituting these and solving for q, gives

$$q = \tfrac{1}{4}\pi \sqrt{\frac{2gh}{f_1 \frac{l_1}{d_1^5} + f_2 \frac{l_2}{d_2^5} + \text{etc.}}}, \quad . \quad . \quad . \quad (60)'$$

in which f_1, f_2, etc., are the friction factors corresponding to the given diameters and velocities in Table XVI.

For example, take the case of two sizes for which the dimensions are

$$d_1 = 2 \text{ feet}, \qquad l_1 = 2800 \text{ feet},$$
$$d_2 = 1.5 \text{ feet}, \qquad l_2 = 2145 \text{ feet}, \qquad h = 127.5 \text{ feet}.$$

Using for f_1 and f_2 the mean value 0.02, and making the substitutions in the formula, there is found

$$q = 27.3 \text{ cubic feet per second,}$$

from which $v_1 = 8.7$ and $v_2 = 15.4$ feet per second.

Now from the table in Art. 74 it is seen that $f_1 = 0.015$ and $f_2 = 0.015$; and repeating the computation,

$$q = 30.1 \text{ cubic feet per second,}$$

which gives $v_1 = 9.60$ and $v_2 = 17.0$ feet per second.

These results are probably as definite as the table of friction factors will allow, but are to be regarded as liable to an uncertainty of several per cent.

To determine the diameter of a pipe which will give the same discharge as the compound one, it is only necessary to replace the denominator in the above value of q by $f\frac{l}{d^5}$, where $l = l_1 + l_2 +$ etc., and d is the diameter required. Taking the values of f as equal, this gives

$$\frac{l}{d^5} = \frac{l_1}{d_1^5} + \frac{l_2}{d_2^5} + \text{etc.}$$

Applying this to the above example, it becomes

$$\frac{4945}{d^5} = \frac{2800}{2^5} + \frac{2145}{1.5^5},$$

from which $d = 1.68$ feet, or about 20 inches.

There seems to be no well-founded reason why a compound pipe should be used, although they have been laid in some cases, the largest size being nearest the reservoir, and the smallest at the outlet end. The above discussion indicates, however, that it is immaterial in what order they be laid, and seems to imply that the greatest advantage results when the size is uniform.

Prob. 105. Compute the discharge when $d_1 = 1.25$, $d_2 = 1.00$, $d_3 = 0.75$, $l_1 = 1200$, $l_2 = 1800$, $l_3 = 800$, and $h = 97.5$ feet.

ARTICLE 83. PIEZOMETER MEASUREMENTS.

Let a piezometer tube be inserted into a pipe at any point D_1, whose distance from the reservoir is l_1 measured along the pipe line. Let A_1D_1 be the vertical depth of this point below the water level of the reservoir; then if the flow be stopped at the end C, the water rises in the tube to the point A_1. But when the flow occurs, the water level in the piezometer stands at some point C_1, and the piezometric height or pressure-head is h_1, or C_1D_1 in the figure. The distance A_1C_1 then represents the velocity-head plus all the losses of head between D_1 and the reservoir. If no losses of head occur except at entrance and in friction, the value of A_1C_1 then is

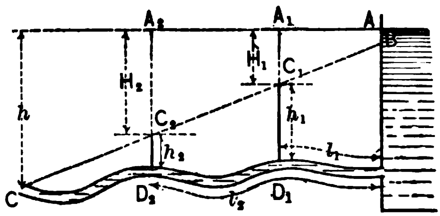


FIG. 56.

$$H_1 = \frac{v^2}{2g} + m\frac{v^2}{2g} + f\frac{l_1}{d}\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad (61)$$

from which the piezometric height can be found when v has been determined by measurement or by computation.

For example, let the total length $l = 3000$ feet, $d = 6$ inches, $h = 9$ feet, and $m = 0.5$. Then, as in Art. 76, there is found

$f = 0.026$ and $v = 1.917$ feet per second. The position of the top of the piezometric column is then given by

$$H_1 = (1.5 + 0.052 l_1) \times 0.05714,$$

and the height of that column is

$$h_1 = A_1 D_1 - H_1 .$$

Thus if $l_1 = 1000$ feet, $H_1 = 3.06$ feet; and if $l_1 = 2000$ feet, $H_1 = 6.03$ feet. If the pipe is so laid that A_1D_1 is 9 feet, the corresponding pressure-heights are then 5.94 and 2.97 feet.

For a second piezometer inserted at D_2 at the distance l_2 from the entrance the value of H_2 is

$$H_2 = \frac{v^2}{2g} + m\frac{v^2}{2g} + f\frac{l_2}{d}\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad (61)$$

From this, subtracting the preceding equation, there is found

$$H_2 - H_1 = f\frac{l_2 - l_1}{d}\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The second member of this formula is the head lost in friction in the length $l_2 - l_1$ (Art. 74), and the first member is the difference of the piezometer elevations. Thus is again proved the principle of Art. 70, that the difference of two piezometer elevations shows the head lost in the pipe between them; in Art. 70 the elevations H_1 and H_2 were measured upward from the datum plane, while here they are measured downward.

By the help of this principle the velocity of flow in a pipe may be approximately determined. A line of levels is run between the points D_1 and D_2, which are selected so that no sharp curves occur between them, and thus the difference $H_2 - H_1$ is found; $l_2 - l_1$, or the length between D_1 and D_2,

is ascertained by careful chaining. Then, from the above formula,

$$v = \sqrt{\frac{2g(H_2 - H_1)d}{f(l_2 - l_1)}}, \quad . \quad . \quad . \quad . \quad . \quad (62)'$$

from which v can be computed by the help of the friction factors in the table of Art. 74. For example, STEARNS, in 1880, made experiments on a conduit pipe 4 feet in diameter under different velocities of flow.* In experiment No. 2 the length $l_2 - l_1$ was 1747.2 feet, and the difference of the piezometer levels was 1.243 feet. Assuming for f the mean value 0.02, and using 32.16 for g, the velocity was

$$v = \sqrt{\frac{64.32 \times 1.243 \times 4}{0.02 \times 1747}} = 3.0 \text{ feet per second.}$$

This velocity in the table of friction factors gives $f = 0.015$ for a 4-foot pipe. Hence, repeating the computation, there is found $v = 3.50$ feet per second; it is accordingly uncertain whether the value of f is 0.015 or 0.014. If the latter value be used there is found

$$v = 3.62 \text{ feet per second.}$$

The actual velocity, as determined by measurement of the water over a weir, was 3.738 feet per second, which shows that the computation is in error about 4 per cent.

The gauging of the flow of a pipe by piezometers is liable to give defective results, partly because the piezometer may not indicate the mean pressure in the pipe owing to an imperfect manner of connection, and partly because the formula for computing the velocity is merely an empirical one. The difference $H_2 - H_1$ in order to be reliable should be taken at

* Transactions of American Society of Civil Engineers, 1885, vol. xiv. p. 1.

points as far apart as possible, and care be taken that no losses of head occur between them except that due to friction. Easy curves give no perceptible loss of head and need not be considered, but obstructions in the pipe or changes in section may render the measurement valueless. When pressure gauges are used, as must be often the case under high heads, care should be taken to test them before making the experiment. The pressure gauges, as generally graduated, give the pressures in pounds per square inch. If then the readings p_1 and p_2 are taken at D_1 and D_2, the pressure-heads in feet are

$$h_1 = 2.304p_1 \quad \text{and} \quad h_2 = 2.304p_2.$$

The vertical distances A_1D_1 and A_2D_2 having been previously determined by levels, the heads H_1 and H_2 are

$$H_1 = A_1D_1 - h_1 \quad \text{and} \quad H_2 = A_2D_2 - h_2,$$

from which $H_2 - H_1$ is known. Or if the vertical fall z between D_1 and D_2 is determined,

$$H_2 - H_1 = h_1 - h_2 + z,$$

which is the loss of head between D_1 and D_2

Prob. 106. At a point 500 feet from the reservoir, and 28 feet below its surface, a pressure gauge reads 10.5 pounds per square inch; at a point 8500 feet from the reservoir and 280.5 feet below its surface, it reads 61 pounds per square inch. Show that the discharge per second is about 6 cubic feet if the pipe be 12 inches in diameter.

ARTICLE 84. THE HYDRAULIC GRADIENT.

The hydraulic gradient is a line which connects the water levels in piezometers placed at intervals along the pipe; or rather, it is the line to which the water levels would rise if piezometer tubes were inserted. In Fig. 56 the line BC is the

hydraulic gradient, and it is now to be shown that for a pipe of uniform size this is approximately a straight line. For a pipe discharging freely into the air, as in Fig. 56, this line joins the outlet end with a point B near the top of the reservoir. For a pipe with submerged discharge, as in Fig. 57, it joins the lower water level with the point B.

Let D_1 be any point on the pipe distant l_1 from the reservoir, measured along the pipe line. The piezometer there placed rises to C_1, which is a point in the hydraulic gradient. The equation of this line with reference to the origin A is given by the formula of the preceding article,

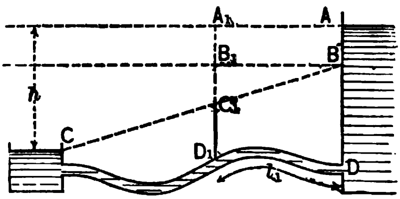


FIG. 57

$$H_1 = (1 + m)\frac{v^2}{2g} + f\frac{l_1}{d}\frac{v^2}{2g},$$

in which H_1 is the ordinate A_1C_1, and l_1 is the abscissa AA_1, provided that the length of the pipe is sensibly equivalent to its horizontal projection. In this equation the term $(1 + m)\frac{v^2}{2g}$ is constant for a given velocity, and is represented in the figure by AB or A_1B_1; the second term varies with l_1, and is represented by B_1C_1. The gradient is therefore a straight line, subject to the provision that the pipe is laid approximately horizontal; which is usually the case in practice, since quite material vertical variations may exist in long pipes without sensibly affecting the horizontal distances.

When the variable point D_1 is taken at the outlet end of the pipe, H_1 becomes the head h, and l_1 becomes the total length l, agreeing with the formula of Art. 76, if the losses of head due to curvature and valves be omitted. When D_1 is

taken very near the inlet end, l becomes zero and the ordinate H_1 becomes AB, which represents the velocity-head plus the loss of head at entrance.

When easy horizontal curves exist, the above conclusions are unaffected, except that the gradient BC is always vertically above the pipe, and therefore can be called straight only by courtesy, although as before the ordinate B_1C_1 is proportional to l_1. When sharp curves exist, the hydraulic gradient is depressed at each curve by an amount equal to the loss of head which there occurs.

If the pipe is so laid that a portion of it rises above the hydraulic gradient as at D_1 in Fig. 58, an entire change of condition generally results. If the pipe be closed at C, all the piezometers stand in the line AA, at the same level as the surface of the reservoir. When the valve at C is opened, the flow at

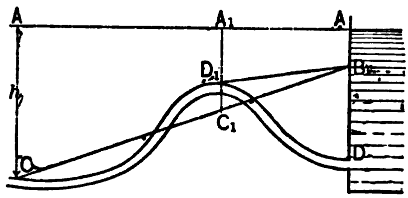


FIG. 58.

first occurs under normal conditions, h being the head and BC the hydraulic gradient. The pressure-head at D_1 is then negative, and represented by D_1C_1. This results in a partial vacuum in that portion of the pipe whereby the continuity of the flow is broken, and as a consequence the pipe from D_1 to C is only partly filled with water. The hydraulic gradient is then shifted to BD_1, the discharge occurs at D_1 under the head A_1D_1, while the remainder of the pipe acts merely as a channel to deliver the flow. It usually happens that this change results in a great diminution of the discharge, so that it has often been necessary to dig up and relay portions of a pipe line which have been inadvertently run above the hydraulic gradient. This trouble can always be avoided by preparing a

profile of the proposed route, and drawing the hydraulic gradient upon it.

When a large part of a pipe lies above the hydraulic gradient it is called a siphon. Conditions sometimes exist which require the construction of siphons, and to insure their successful action pumps must be attached near the highest elevations, which may be occasionally operated to remove the air that has accumulated, and which would otherwise cause the flow to diminish and ultimately to cease.

The pressure-head, or piezometer height h_1, at any point of the pipe can be computed if the velocity of flow is known, as also the depth H of that point below the water surface in the reservoir. In the above figures the ordinate A_1D_1 is the depth H. Then

$$h_1 = H - \left(1 + m + f\frac{l_1}{d}\right)\frac{v^2}{2g},$$

in which v must be known by measurement or be computed by the method of Art. 76 from the total length l and the given head h. This may be put into a simpler form by substituting for v its value in terms of l and h, which gives

$$h_1 = H - \frac{1 + m + f\dfrac{l_1}{d}}{1 + m + f\dfrac{l}{d}}h; \quad . \quad . \quad . \quad . \quad . \quad (63)$$

or for long pipes, where $1 + m$ may be neglected,

$$h_1 = H - \frac{l_1}{l}h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)'$$

This formula, indeed, can be directly derived from the above figures by similar triangles, taking the point B as coincident with A, which for long pipes is allowable, since AB is very small (Art. 80).

The above discussion shows that it is immaterial where the pipe enters the reservoir, provided that it enters below the hydraulic gradient point B. It is also not to be forgotten that the whole investigation rests on the assumption that the lengths l_1 and l are sensibly equal to their horizontal projections.

Prob. 107. A pipe 3 inches in diameter discharges 538 cubic feet per hour under a head of 12 feet. At a distance of 300 feet from the reservoir the depth of the pipe below the water surface in the reservoir is 4.5 feet. Compute the probable pressure-head at this point. Ans. $-$ 0.2 feet.

ARTICLE 85. A PIPE WITH A NOZZLE.

Water is often delivered through a nozzle in order to perform work upon a motor or for the purposes of hydraulic mining, the nozzle being attached to the end of a pipe which brings the flow from a reservoir. In such a case it is desirable that the pressure at the entrance to the nozzle should be as great as possible, and this will be effected when the loss of head in the pipe is as small as possible. The pressure column in a piezometer, supposed to be inserted at the end of the pipe, as shown at C_1D_1 in Fig. 59, measures the pressure-head there acting, and the height A_1C_1 measures the lost head plus the velocity-head, the latter being very small.

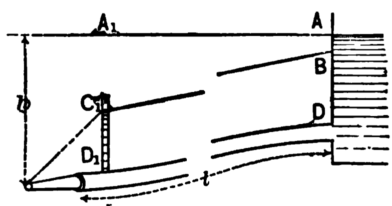


FIG. 59.

Let h be the total head on the end of the nozzle, l_1 its length, d_1 its diameter, and v_1 the velocity of discharge at the small end. Let l, d, and v be the corresponding quantities for the pipe. Then the effective velocity-head of the issuing stream is $\frac{v_1^2}{2g}$, and the lost head is $h - \frac{v_1^2}{2g}$. This lost head consists of

several parts—that lost at the entrance D_1; that lost in friction in the pipe; that lost in curves and valves, if any; and lastly, that lost in the nozzle. Thus,

$$h - \frac{v_1^2}{2g} = m\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g} + n\frac{v^2}{2g} + m_1\frac{v_1^2}{2g}.$$

Here m is determined by Art. 73, f by Art. 74, n by Art. 75, and m_1 is to be found from the coefficient of velocity of the nozzle (Art. 63) in the same manner as m. If, for instance, c_1 for the nozzle is 0.98, then

$$m_1 = \left(\frac{1}{0.98^2} - 1\right) = 0.04;$$

and for a perfect nozzle m_1 would be zero. The value of m_1 includes all losses of head in the nozzle, as m does in the entrance tube, so that the length l_1 need not be considered.

The velocities v and v_1 are inversely as the areas of the corresponding sections, whence

$$v_1 = v\frac{d^2}{d_1^2}.$$

Inserting this in the above expression, and solving for v, gives the formula

$$v = \sqrt{\frac{2gh}{m + n + f\frac{l}{d} + (1 + m_1)\frac{d^4}{d_1^4}}}, \quad . \quad . \quad . \quad (64)$$

from which v can be computed by the tentative method explained in Art. 76. This equation, in connection with the preceding, shows that the greatest velocity v_1 obtains when d is as large as possible compared to d_1. As the object of a nozzle is to utilize either the velocity or the energy of the water, a large pipe and a small nozzle should hence be employed to give the best result, and this is attained when the velocity v_1 is nearly equal to $\sqrt{2gh}$.

As a numerical example, the effect of attaching a nozzle to the pipe whose discharge was computed in Art. 77 will be considered. There $l = 500$, $d = 0.25$, and $h = 4$ feet; $m = 0.5$, $n = 0$, $v = 2.15$ feet; and $q = 0.106$ cubic feet, per second. Now let the nozzle be one inch in diameter at the small end, or $d_1 = 0.0833$ feet and $c_1 = 0.98$, whence $m_1 = 0.041$. Using $f = 0.029$, the velocity in the pipe is

$$v = \sqrt{\frac{2 \times 32.16 \times 4}{0.5 + 0.029 \times 500 \times 4 + 1.041 \times 81}};$$

whence $v = 1.35$ feet per second. The effect of the nozzle, therefore, is to reduce the velocity, owing to the loss of head which it causes. The velocity of flow from the nozzle is

$$v_1 = 1.35 \times 9 = 12.15 \text{ feet per second};$$

and the discharge per second is

$$q = 0.7854 \times 0.25^2 \times 1.35 = 0.066 \text{ cubic feet}.$$

which is about 40 per cent less than that of the pipe before the nozzle was attached. The nozzle, however, produces a marvellous effect in increasing the energy of the discharge; for the velocity-head corresponding to 2.15 feet per second is only 0.072 feet, while that corresponding to 12.15 feet per second is 2.30 feet, or about 32 times as great. As the total head is 4 feet, the efficiency of the stream issuing from the nozzle is about 57 per cent.

If the pressure-head h_1 at the entrance of the nozzle be observed, either by a piezometer or by a pressure gauge, the velocity of discharge can be computed by the formula

$$v_1 = \sqrt{\frac{2gh_1}{1 + m_1 - \frac{d_1^4}{d^4}}},$$

whose demonstration is given in Art. 63. If both h_1 and v_1 be

measured, this formula furnishes the means of computing m_1, or the loss of head caused by the nozzle.*

Prob. 108. Compute the velocity and effective velocity-head for the above pipe and nozzle, but taking the head h as 16 feet.

ARTICLE 86. HOUSE SERVICE PIPES.

A service pipe which runs from a street main to a house is connected to the former at right angles, and usually by a "ferrule" which is smaller in diameter than the pipe itself.

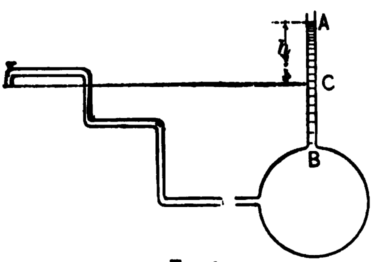


FIG. 60.

The loss of head at entrance is hence larger than in the cases before discussed, and m should probably be taken as at least equal to unity. The pipe, if of lead, is frequently carried around sharp corners by curves of small radius; if of iron, these curves are formed by pieces forming a quadrant of a circle into which the straight parts are screwed, the radius of the centre line of the curve being but little larger than the radius of the pipe, so that each curve causes a loss of head equal nearly to double the velocity-head (Art. 75). For new clean pipes the loss of head due to friction may be estimated by the rules of Art. 74.

A water main should be so designed that a certain minimum pressure-head h_1 exists in it at times of heaviest draught. This pressure-head may be represented by the height of the piezometer column AB, which would rise in a tube supposed to be inserted in the main, as in Fig. 60. The head h which causes the flow in the pipe is then the difference in level between

* For experimental formulas for flow through hose and nozzles, see a paper by WESTON in Transactions American Society Civil Engineers, 1884, vol. xiii. p. 376.

the top of this column and the end of the pipe, or AC. Inserting for h this value, the formulas of Arts. 76 and 77 may be applied to the investigation of service pipes, in the manner there illustrated. As the sizes of common house-service pipes are regulated by the practice of the plumbers and by the market sizes obtainable, it is not often necessary to make computations regarding them.

The velocity of flow in the main has no direct influence upon that in the pipe, since the connection is made at right angles. But as that velocity varies, owing to the varying draught upon the main, the pressure-head h_1 is subject to continual fluctuations. When there is no flow in the main, the piezometer column rises until its top is on the same level as the surface of the reservoir; in times of great draught it may sink below C, so that no water can be drawn from the service pipe.

The detection and prevention of the waste of water by consumers is a matter of importance in cities where the supply is limited and where meters are not in use. Of the many methods devised to detect this waste, one by the use of piezometers may be noticed, by which an inspector without entering a house may ascertain whether water is being drawn within, and the approximate amount per second. Let M be the street main from which a service pipe MOH runs to a house H. At the edge of the sidewalk a tube OP is connected to the service pipe, which has a three-way cock at O, which can be turned from above. The inspector, passing on his rounds in the night-time, attaches a pressure gauge at P and turns the cock O so as to shut off the water from the house and allow the full pressure of the main p_1 to be registered. Then he turns the cock so that the water may flow into the house, while it also rises in OP and registers the pressure p_2. Then if p_2 is less than p_1 it is certain that a

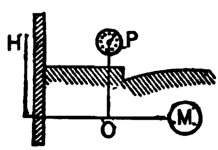


FIG. 61.

waste is occurring within the house, and the amount of this may be approximately computed if desired, in the manner indicated in Art. 70, and the consumer be fined accordingly.*

Prob. 109. Describe a water-pressure regulator to be placed between the main and the house so that the pressure in the service pipes may never exceed a given quantity—say 40 pounds per square inch.

ARTICLE 87. A WATER MAIN.

The simplest case of the distribution of water is that where a single main is tapped by a number of service pipes near its end, as shown in Fig. 62. In designing such a main the principal consideration is that it should be large enough so that the pressure-head h_1, when all the pipes are in draught, shall be amply sufficient to deliver the water into the highest houses along the line. FANNING recommends that this pressure-head in commercial and manufacturing districts should not be less than 150 feet, and in suburban districts not less than 100 feet. The height H to the surface of the water in the reservoir will always be greater than h_1, and the pipe is to be so designed that the losses of head may not reduce h_1 below the limit assigned. The head h to be used in the formulas is the difference $H - h_1$. The discharge per second q being known or assumed, the problem is to determine the diameter d of the main.

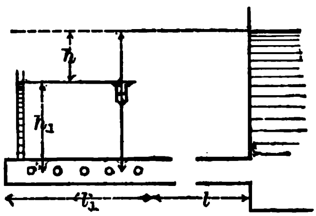


FIG. 62.

A strict theoretical solution of even this simple case leads to very complicated calculations, and in fact cannot be made without knowing all the circumstances regarding each of the service pipes. Considering that the result of the computation

* This briefly describes CHURCH's water-waste indicator.

is merely to enable one of the market sizes to be selected, it is plain that great precision cannot be expected, and that approximate methods may be used to give a solution entirely satisfactory. It will then be assumed that the service pipes are connected with the main at equal intervals, and that the discharge through each is the same under maximum draught. The velocity v in the main then decreases, and becomes 0 at the dead end. The loss of head per linear foot in the length l_1 (Fig. 62) is hence less than in l. To estimate this, let v_1 be the velocity at a distance x from the dead end; then

$$v_1 = \frac{x}{l_1} v.$$

The loss of head in friction in the length δx is

$$\delta h'' = f \frac{\delta x}{d} \frac{v_1^2}{2g} = f \frac{x^2}{dl_1^2} \frac{v^2}{2g} \delta x;$$

and hence between the limits 0 and l_1 that loss is

$$h'' = f \frac{l_1}{3d} \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

provided that f remains constant. This is really not the case, but no material error is thus introduced, since f must be taken larger than the tabular values in order to allow for the deterioration of the inner surface of the main. The loss of head in friction for a pipe which discharges uniformly along its length may therefore be taken at one-third of that which occurs when the discharge is entirely at the end.

Now neglecting the loss of head at entrance and the effective velocity-head of the discharge, the total head h is entirely consumed in friction, or

$$h = f \frac{l}{d} \frac{v^2}{2g} + f \frac{l_1}{3d} \frac{v^2}{2g}.$$

Placing in this for v its value in terms of the total discharge q, and solving for d, gives

$$d^5 = (l + \tfrac{1}{3}l_1)\frac{16fq^2}{2g\pi^2 h}.$$

This is the same as the formula of Art. 80, except that l has been replaced by $l + \frac{1}{3}l_1$. The diameter in feet then is

$$d = 0.479\,(l + \tfrac{1}{3}l_1)^{\frac{1}{5}}\left(\frac{fq^2}{h}\right)^{\frac{1}{5}},$$

as in the case of long pipes.

For example, consider a village consisting of a single street, whose length $l_1 = 3000$ feet, and upon which there are 100 houses, each furnished with a service pipe. The probable population is then 500, and taking 100 gallons per day as the consumption per capita, this gives the average discharge per second

$$q = \frac{500 \times 100}{7.48 \times 3600 \times 24} = 0.0774 \text{ cubic feet};$$

and as the maximum draught is often double of the average, q will be taken as 0.15 cubic feet per second. The length l to the reservoir is 4290 feet, whose surface is 90.5 feet above the dead end of the main, and it is required that under full draught the pressure-head in the main shall be 75 feet. Then $h = 90.5 - 75 = 15.5$ feet, and taking $f = 0.03$ in order to be on the safe side, the formula gives

$$d = 0.36 \text{ feet} = 4.3 \text{ inches}.$$

Accordingly a four-inch pipe is nearly large enough to satisfy the imposed conditions.

To consider the effect of fire service upon the diameter of the main, let there be four hydrants placed at equal intervals along the line l_1, each of which is required to deliver 20 cubic feet per minute under the same pressure-head of 75 feet. This gives a discharge 1.33 cubic feet per second, or, in total,

$q = 1.33 + 0.15 = 1.5$ cubic feet. Inserting this in the formula, and using for f the same value as before,

$$d = 0.897 \text{ feet} = 10.8 \text{ inches}.$$

Hence a ten-inch pipe is at least required to maintain the required pressure when the four hydrants are in full draught at the same time with the service pipes.

Prob. 110. Compute the velocity v and the pressure-head h_1 for the above example, if the main be 10 inches in diameter and the discharge 1.5 cubic feet per second.

ARTICLE 88. A MAIN WITH BRANCHES.

In Fig. 63 is shown a main of length l and diameter d, having two branches with lengths l_1 and l_2, and diameters d_1

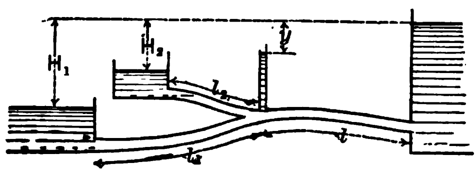


FIG. 63.

and d_2. These being given, as also the heads H_1 and H_2 under which the flow occurs, it is required to find the discharges q_1 and q_2. Let v, v_1, and v_2 be the corresponding velocities; then for long pipes, in which all losses except those due to friction may be neglected,

$$H_1 - y = f_1 \frac{l_1}{d_1} \frac{v_1^2}{2g},$$

$$H_2 - y = f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g},$$

where y is the difference in level between the reservoir surface and the water level in a piezometer supposed to be inserted at the junction. This y is the friction-head consumed in the large main, or

$$y = f \frac{l}{d} \frac{v^2}{2g}.$$

Inserting this in the two equations, and placing for the velocities their values in terms of the discharges, they become

$$\left.\begin{aligned}\frac{2g\pi^2}{16}H_1 &= f\frac{l}{d^5}(q_1 + q_2)^2 + f_1\frac{l_1}{d_1^5}q_1^2 \\ \frac{2g\pi^2}{16}H_2 &= f\frac{l}{d^5}(q_1 + q_2)^2 + f_2\frac{l_2}{d_2^5}q_2^2\end{aligned}\right\}, \quad . \quad . \quad (66)$$

from which q_1 and q_2 are best obtained by trial; although by solution the value of each may be directly expressed in terms of the given data, the expressions are too complicated for general use.

When it is required to determine the diameters from the given lengths, heads, and discharges, there are three unknown quantities, d, d_1, d_2, to be found from only two equations, and the problem is indeterminate. If, however, d be assumed, values of d_1 and d_2 may be found; and as d may be taken at pleasure, it appears that an infinite number of solutions is possible. Another way is to assume a value of y, corresponding to a proper pressure-head at the junction; then the diameters are directly found from the usual formula for long pipes,

$$d = 0.479\left(\frac{flq^2}{h}\right)^{\frac{1}{5}},$$

in which h is replaced by y for the large main, and by $H_1 - y$ and $H_2 - y$ for the smaller ones, q for the first being $q_1 + q_2$, and for the others q_1 and q_2 respectively.

A water-supply system consists of a principal main with many sub-mains as branches. In designing these the quantities of water to be furnished are assumed from the present and probable future population, which in small towns requires from 40 to 100 gallons per capita per day, and in large cities from 100 to 150 gallons. This should be furnished under heads sufficient to raise the water into the highest houses, as also for use in

cases of fire. As the problem of computing the diameters from the given data is indeterminate, it will probably be as well to assume at the outset the sizes for the principal mains. The velocities corresponding to the given quantities and the assumed sizes are then computed, and from these the pressure-heads at a number of points are found. If these are not satisfactory, other sizes are to be taken and the computation be repeated. The successful design will be that which will furnish the required quantities under proper pressures with the least expenditure.

Prob. 111. In Fig. 63 let $q_1 = 0.5$ and $q_2 = 0.4$ cubic feet; $H_1 = 140$ and $H_2 = 125$ feet; $l_1 = 3810$, $l_2 = 2455$, and $l = 12\,314$ feet. If d_1 equals d_2 find the values of d and d_1, and also the pressure-head at the junction if its depth below the reservoir level is 108 feet.

ARTICLE 89. PUMPING THROUGH PIPES.

When water is pumped through a pipe from a lower to a higher level, the power of the pump must be sufficient not only to raise the required amount in a given time, but also to overcome the various resistances to flow. The head due to the resistances is thus a direct source of loss, and it is desirable that the pipe be so arranged as to render this as small as possible.

Let w be the weight of a cubic foot of water and q the quantity raised per second through the height H, which, for example, may be the difference in level between a canal C and a reservoir R, as in Fig. 64. The useful work done by the pump in each second is wqH. Let h' be the head lost in entering the pipe at the canal, h'' that lost in friction in the pipe, and h''' all other losses of head, such as those caused by

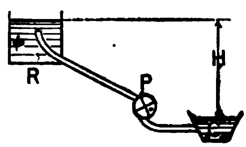


FIG. 64.

curves, valves, and by resistances in passing through the pump cylinders. Then the total work performed by the pump per second is

$$k = wqH + wq(h' + h'' + h'''). \quad . \quad . \quad . \quad (67)$$

Inserting for the lost heads their values, this becomes

$$k = wqH + wq\left(m + f\frac{l}{d} + n\right)\frac{v^2}{2g}. \quad . \quad . \quad . \quad (67)'$$

In order, therefore, that the losses of work may be as small as possible, the velocity of flow through the pipe should be low; and this is to be effected by making the diameter of the pipe large.

For example, let it be required to determine the horse-power of a pump to raise 1 200 000 gallons per day through a height of 230 feet, when the diameter of the pipe is 6 inches and its length 1400 feet. The discharge per second is

$$q = \frac{1\,200\,000}{7.481 \times 24 \times 3600} = 1.86 \text{ cubic feet,}$$

and the velocity of flow is

$$v = \frac{1.86}{0.7854 \times 0.5^2} = 9.47 \text{ feet per second.}$$

The probable head lost at entrance into the pipe is

$$h' = 0.5\frac{v^2}{2g} = 0.5 \times 1.39 = 0.7 \text{ feet.}$$

When the pipe is new and clean the friction factor f is about 0.020, as shown by Table XVI; then the loss of head in friction is

$$h'' = 0.020 \times \frac{1400}{0.5} \times 1.39 = 77.8 \text{ feet.}$$

The other losses of head depend upon the details of the valve and pump cylinder; if these be such that $n = 4$, then

$$h''' = 4 \times 1.39 = 5.6 \text{ feet.}$$

The total losses of head hence are

$$h' + h'' + h''' = 84.1 \text{ feet.}$$

The work to be performed per second by the pump now is

$$k = 62.5 \times 1.86(230 + 84.1) = 36\,510 \text{ foot-pounds,}$$

and the horse-power expended is

$$\overline{HP} = \frac{36\,510}{550} = 66.4.$$

If there were no losses in friction and other resistances the work done would be simply

$$k = 62.5 \times 1.86 \times 230 = 26\,740 \text{ foot-pounds,}$$

and the corresponding horse-power would be

$$\overline{HP} = \frac{26\,740}{550} = 48.6.$$

Accordingly 17.8 horse-power is wasted in injurious resistances.

For the same data let the 6-inch pipe be replaced by one 14 inches in diameter. Then, proceeding as before, the velocity of flow is found to be 1.80 feet per second, the head lost at entrance 0.03 feet, the head lost in friction 1.23 feet, and that lost in other ways 0.20 feet. The total losses of head are thus only 1.46 feet, as against 84.1 feet for the smaller pipe, and the horse-power required is 48.9, which is but little greater than the theoretic power. The great advantage of the larger pipe is thus apparent, and by increasing its size to 18 inches the losses of head may be reduced so low as to be scarcely appreciable in comparison with the useful head of 230 feet.

A pump is often used to force water directly through the

mains of a water-supply system under a designated pressure. The work of the pump in this case consists of that required to maintain the pressure and that required to overcome the frictional resistances. Let h_1 be the pressure-head to be maintained at the end of the main, and z the height of the main above the level of the river from which the water is pumped; then $h_1 + z$ is the head H, which corresponds to the useful work of the pump, and, as before,

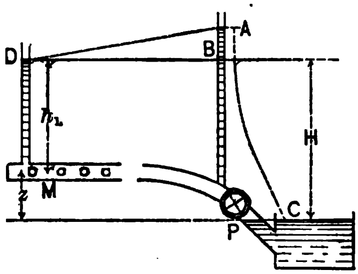


FIG. 65.

$$k = wqH + wq(h' + h'' + h''').$$

To reduce these injurious heads to the smallest limits the mains should be large in order that the velocity of flow may be small. In Fig. 65 is shown a symbolic representation of the case of pumping into a main, P being the pump, C the source of supply, and DM the pressure-head which is maintained upon the end of the pipe during the flow. At the pump the pressure-head is AP, so that AD represents the hydraulic gradient for the pipe from P to M. The total work of the pump may then be regarded as expended in lifting the water from C to A, and this consists of three parts, corresponding to the heads CM or z, MD or h_1, and AB or $h' + h'' + h'''$, the first overcoming the force of gravity, the second delivering the flow under the required pressure, while the last is transformed into heat in overcoming friction and other resistances. In this direct method of water supply a standpipe, AP, is often erected near the pump, in which the water rises to a height corresponding to the required pressure, and which furnishes a supply when a temporary stoppage of the pumping engine occurs.

Prob. 112. Compute the horse-power of a pump for the fol-

lowing data, neglecting all resistances except those due to friction: $q = 1.5$ cubic feet per second, which is distributed uniformly over a length $l_1 = 3000$ feet, the remaining length of the pipe being 4290 feet; $d = 10$ inches, $h_1 = 75$ feet, $z = 0.4$ feet.

ARTICLE 90. LEATHER AND RUBBER HOSE.

The losses of head in friction are greater in leather and rubber hose than in clean iron pipes, especially so at low velocities. The following are values of the friction factor f which have been deduced from experiments made by ELLIS,* on hose $2\frac{1}{2}$ inches in diameter, to which are added the values for an iron pipe of the same size:

For velocity v =	3	4	6	10	15	25 feet,
f for leather hose =	0.095	0.064	0.043	0.033	0.030	0.029
f for rubber hose =	0.045	0.033	0.027	0.025	0.026	0.027
f for iron pipe =	0.027	0.026	0.025	0.023	0.022	

By the help of this table computations may be made on the pumping of water through hose for delivery in fire streams or for other purposes, in the same manner as for pipes.

The loss of head in a long hose becomes so great even under moderate velocities as to consume a large proportion of the pressure exerted by the hydrant or steamer. For example, let this primitive pressure be 122 pounds per square inch, corresponding to a head of 281 feet, and let it be required to find the pressure-head in the $2\frac{1}{2}$-inch leather hose at 1000 feet distance, when a nozzle is used, which discharges 153 gallons per minute, the hose being laid horizontal. In cubic feet per second the discharge is

$$q = \frac{153}{7.48 \times 60} = 0.341,$$

* G. A. ELLIS, Fire Streams (Springfield, 1878).

and the velocity in the hose is accordingly found to be

$$v = \frac{q}{\frac{1}{4}\pi d^2} = 10.0 \text{ feet per second}$$

Hence the loss of head in friction is

$$h'' = f\frac{l}{d}\frac{v^2}{2g} = 246 \text{ feet,}$$

and consequently the pressure-head at the entrance to the nozzle is

$$h_1 = 281 - 246 = 35 \text{ feet,}$$

which corresponds to about 15 pounds per square inch. The remedy for this great reduction of pressure is to employ a smaller nozzle, thus decreasing the discharge and the velocity in the hose; but if both head and quantity of discharge are desired they can only be secured either by an increase of pressure at the steamer or by the use of a larger hose.

Prob. 113. When the pressure gauge at the steamer indicates 83 pounds per square inch, a gauge on the leather hose 800 feet distant reads 25 pounds. Compute the value of the friction factor f, the discharge per minute being 121 gallons.
Ans. 0.036.

ART. 91. LAMPE'S FORMULA.

There have been made many attempts to express the mean velocity v without the use of a factor or coefficient of friction. That this can be empirically done, within the range of experimental results, is plain by observing that the values of f in Table XVI show a regular variation with the diameter d. For long pipes f is then a function of d and v, or a function of d, h, and l. The simplest expression of the relation between these quantities is

$$v = \alpha d^{\beta}\left(\frac{h}{l}\right)^{\gamma},$$

in which α, β, and γ are empirical constants. The investigations of LAMPE have determined probable values for these constants, giving

$$v = 77.7 d^{0.694}\left(\frac{h}{l}\right)^{0.555}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

in which d, h, and l are to be taken in feet, and v will be found in feet per second, This formula is only applicable to long circular pipes with surfaces clean or in fair condition.

From this formula the discharge q may be expressed

$$q = 61.0 d^{2.694}\left(\frac{h}{l}\right)^{0.555}, \quad . \quad . \quad . \quad . \quad . \quad (68)'$$

and the diameter required to discharge a given quantity is

$$d = 0.217 q^{0.371}\left(\frac{l}{h}\right)^{0.206}. \quad . \quad . \quad . \quad . \quad . \quad (68)''$$

By the use of these formulas all of the preceding problems concerning long pipes may be directly solved without the use of the tables of friction factors. They show that the discharging capacity of long pipes varies about as the 2.7 power of the diameter (Art. 80).

As an example, let it be required to find the diameter of a pipe which is to discharge 177 300 gallons per hour, its length being 75 000 feet and the head 135 feet. Here

$$q = \frac{177\,300}{3600 \times 7.481} = 6.583 \text{ cubic feet,}$$

and $$\frac{l}{h} = \frac{75\,000}{135} = 555.6;$$

whence by the formula

$$d = 0.217(6.583)^{0.371}(555.6)^{0.206},$$

which gives

$$d = 1.61 \text{ feet} = 19.3 \text{ inches,}$$

so that a 20-inch pipe should be selected.

Prob. 114. Solve Problems 102 and 103 by the use of the above formulas.

ARTICLE 92. VERY SMALL PIPES.

The preceding investigations and rules apply to pipes greater than about 0.5 inches in diameter, and are not valid for very small pipes. The laws of discharge in these are not understood from a theoretical basis, but experiments made by POISEUILLE in 1843, in order to study the phenomena of the flow of blood in veins and arteries, have settled beyond question that they are materially different from those which govern large pipes at ordinary velocities. His investigations proved that for pipes whose diameters are less than about 0.7 millimeters or 0.03 inches, the velocity is expressed by the simple relation

$$v = \alpha \frac{hd^2}{l},$$

in which α is a factor nearly constant at a given temperature. The velocity then varies directly with the head and with the square of the diameter, and inversely with the length. It is here supposed that the pipe is long, so that losses of head due to entrance may be neglected.

Later researches indicate that these laws are also true for large pipes, provided the velocity be small; and that for a given pipe there is a certain critical velocity at which the law changes, and beyond which

$$v = \beta \sqrt{\frac{hd}{l}},$$

as for the case of common pipes. This critical point appears to be that at which the filaments cease to move in parallel lines, and pass in sinuous paths from one side of the pipe to the other. For a very small pipe the velocity may be high before this point is reached; for a large pipe it happens at very low velocities.

In Art. 74 it was mentioned that the frictional resistances in a pipe consist of those along the inner surface, and of those met among the particles in their sinuous motion. Since in small pipes the latter do not exist, it appears from POISEUILLE'S formula that the head lost in friction along the inner surface may be expressed by

$$h'' = \frac{lv}{\alpha d^2}.$$

Now if the law were known which governs the loss in internal friction it might be possible to add this to the preceding, and thus obtain an expression for loss of head in which the friction factor would be a quantity dependent only upon the nature of the surface. Thus far, however, efforts in this direction have not been practically successful.

The effect of temperature on the flow has not been considered in the previous articles, and in fact but little is known regarding it, except that a very slight increase in discharge is probable for a high rise in the temperature of the water. For very small pipes, however, POISEUILLE found that a marked increase in velocity and discharge resulted, the value of α being about twice as great at 45° Centigrade as at 10°.

Prob. 115. The value of α for small pipes is about 184 when h, d, l, and v are in millimeters, and the flow occurs at 10° Centigrade. Find its value when the foot is the unit of measure.

CHAPTER VIII.

FLOW IN CONDUITS AND CANALS.

ARTICLE 93. DEFINITIONS.

Water is often conveyed from place to place in artificial channels, such as troughs, aqueducts, ditches, and canals, there being no head to cause the flow except that due to the slope. The word conduit will be used as a general term for a channel lined with timber, mortar, or masonry, and will also include metal pipes, troughs, and sewers. Conduits may be either open as in the case of troughs, or closed as in sewers and most aqueducts. Streams flow in natural channels eroded in the earth, and include small brooks as well as the largest rivers. Most of the principles relating to conduits and canals apply also to streams, and the word channel will be used as applicable to all classes.

The wetted perimeter of the cross-section of a channel is that part of its boundary which is in contact with the water. Thus, if a circular sewer of diameter d be half full of water the wetted perimeter is $\frac{1}{2}\pi d$. In this chapter the letter p will designate the wetted perimeter.

The hydraulic radius of a water cross-section is its area divided by its wetted perimeter. Let a be the area and r the hydraulic radius; then

$$r = \frac{a}{p}.$$

The letter r is of frequent occurrence in formulas for the flow

in channels; it is a linear quantity, which is always expressed in the same unit as p. It is also frequently called the hydraulic depth or hydraulic mean depth, because for a shallow section its value is but little less than the mean depth of the water. Thus in Fig. 66, if b be the breadth on the water surface, the mean depth is $a \div b$, and the hydraulic radius is $a \div p$; and these are nearly equal, since p is but slightly larger than b.

FIG. 66.

The hydraulic radius of a circular cross-section filled with water is one-fourth of the diameter; thus:

$$r = \frac{a}{p} = \frac{\frac{1}{4}\pi d^2}{\pi d} = \tfrac{1}{4}d.$$

The same value is also applicable to a circular section half filled with water, since then both area and wetted perimeter are one-half their former values.

The slope of the water surface in the longitudinal section, designated by the letter s, is the ratio of the fall h to the length l in which that fall occurs, or

$$s = \frac{h}{l}.$$

The slope is hence expressed as an abstract number, which is independent of the system of measures employed. To determine its value with precision h must be obtained by referring the water level at each end of the line to a bench mark by the help of a hook gauge or other accurate means, the benches being connected by level lines run with care. The distance l is measured along the inclined channel, and it should be of considerable length in order that the relative error in h may not be large.

If there be no slope, or $s = 0$, there can be no flow. But if there be even the smallest slope the force of gravity furnishes a component acting down the inclined surface, and motion ensues. The velocity of flow evidently increases with the slope.

The flow in a channel is said to be permanent when the same quantity of water per second passes through each cross-section. If an empty channel be filled by admitting water at its upper end the flow is at first non-permanent or variable, for more water passes through one of the upper sections per second than is delivered at the lower end. But after sufficient time has elapsed the flow becomes permanent; when this occurs the mean velocities in different sections are inversely as their areas (Art. 19).

Uniform flow is that particular case of permanent flow where all the water cross-sections are equal, and the slope of the water surface is parallel to that of the bed of the channel. If the sections vary the flow is said to be non-uniform, or variable, although the condition of permanency is still fulfilled. In this chapter only the case of uniform flow will be discussed.

The velocities of different filaments in a channel are not equal, as those near the wetted perimeter move slower than the central ones owing to the retarding influence of friction. The mean of all the velocities of all the filaments in a cross-section is called the mean velocity v. Thus if v', v'', etc., be velocities of different filaments,

$$v = \frac{v' + v'' + \text{etc.}}{n}, \quad . \quad . \quad . \quad . \quad . \quad (69)$$

in which n is the number of filaments. Let a be the area of the cross-section and a' that of one of the elementary filaments; then $n = \frac{a}{a'}$, and hence

$$av = a'(v' + v'' + \text{etc.}).$$

But the second member is the discharge q. Therefore the mean velocity may be also determined by the relation

$$v = \frac{q}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)'$$

The filaments which are here considered are in part imaginary, for experiments show that there is a constant sinuous motion of particles from one side of the channel to the other. The best definition for mean velocity hence is, that it is a velocity which multiplied by the area of the cross-section gives the discharge, or $v = q \div a$.

Prob. 116. Compute the hydraulic radius of a rectangular trough whose width is 4.4 feet and depth 2.2 feet.

Prob. 117. Compute the mean velocity in a circular sewer of 4 feet diameter when it is half filled and discharges 120 gallons per second.

ARTICLE 94. FORMULA FOR MEAN VELOCITY.

When all the wetted cross-sections of a channel are equal, and the water is neither rising nor falling, having attained a condition of permanency, the flow is said to be uniform. This is the case in a conduit or canal of constant size and slope whose supply does not vary. The same quantity of water per second then passes each cross-section, and consequently the mean velocity in each section is the same. This uniformity of flow is due to the resistances along the interior surface of the channel, for were it perfectly smooth the force of gravity would cause the velocity to be accelerated. The entire energy of the water due to the fall h is hence expended in overcoming frictional resistances along the length l. Let W be the weight of water per second which passes any cross-section, F the force of friction or resistance per square foot of the interior surface of the channel, p the wetted perimeter, and v the mean veloci-

ty. Now assuming that the friction is uniform over the entire inner surface whose area is pl, the total resisting force is Fpl, and again assuming that the velocity along the surface is the same as v, the total resisting work is $Fplv$. Hence

$$Fplv = Wh.$$

But the value of W is wav where a is the area of the cross-section, and w is the weight of a cubic unit of water; accordingly,

$$Fpl = wah,$$

or

$$F = w\frac{ah}{pl}.$$

Here $\frac{a}{p}$ is the hydraulic radius r, and $\frac{h}{l}$ is the slope s, and the value of F is

$$F = wrs.$$

This is an approximate expression for the resisting force of friction per square foot of the interior surface of the channel.

In order to establish a formula for mean velocity the value of F must be expressed in terms of v, and this can only be done by studying the results of experiments. These indicate that F is approximately proportional to the square of the mean velocity. Therefore, if c be a constant,

$$v = c\sqrt{rs}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

This is an empirical expression for the law of variation of the mean velocity with the hydraulic radius and slope of the channel. The quantity c is a coefficient which varies with the degree of roughness of the bed and with other circumstances. It is the object of the following articles to state values of c for different classes of conduits and canals.

Another method of establishing the above formula is simi-

lar to that used in Art. 74 for pipes. The total head h represents the loss of head in friction; this should vary directly with p and l, and it should vary inversely with a, because for a given wetted perimeter the friction will be the least for the largest a. It should also vary as the square of the velocity. Hence

$$h = f' \frac{pl}{a} \frac{v^2}{2g},$$

in which f' is an abstract number depending upon the character of the surface. From this the value of v is

$$v = \sqrt{\frac{2gah}{f'pl}} = c\sqrt{rs}, \quad . \quad . \quad . \quad . \quad . \quad (70)'$$

in which c is the square root of $2g \div f'$. Notwithstanding these reasonings the formula cannot be called rational; it is merely an empirical expression whose basis is experiment.

To determine values of the coefficient c the quantities v, r, and s are measured for particular cases, and then c is computed. To find r and s linear measurements are alone required. To determine v the flow must be gauged either in a measuring vessel or by an orifice or weir, or, if the channel be large, by floats or other indirect methods described in the next chapter. It being a matter of great importance to establish a satisfactory formula for mean velocity, thousands of such gaugings have been made, and from the records of these the values of the coefficients have been deduced. It is found that c lies between 30 and 160 when v and r are expressed in feet, and that its value is subject to variation, not only with the character of the surface, but also with the hydraulic radius and slope.

Prob. 118. Compute the value of c for a circular masonry conduit 4 feet in diameter which delivers 29 cubic feet per second when running half full, its slope or grade being 1.5 feet in 1000 feet. Ans. 119.

ARTICLE 95. CIRCULAR CONDUITS, FULL OR HALF FULL.

When a circular conduit of diameter d runs either full or half full of water the hydraulic radius is $\frac{1}{4}d$, and the formula for mean velocity is

$$v = c\sqrt{rs} = c \cdot \tfrac{1}{2}\sqrt{ds}.$$

The velocity can then be computed when c is known, and for this purpose the following table gives SMITH'S values of c for

TABLE XVII. COEFFICIENTS FOR CIRCULAR CONDUITS.

Diameter in Feet.	Velocity in Feet per Second.						
	1	2	3	4	6	10	15
1.	96	104	109	112	116	121	124
1.5	103	111	116	119	123	129	132
2.	109	116	121	124	129	134	138
2.5	113	120	125	128	133	139	143
3.	117	124	128	132	136	143	147
3.5	120	127	131	135	139	146	151
4.	123	130	134	137	142	150	155
5.	128	134	139	142	147	155	
6.	132	138	142	145	150		
7.	135	141	145	149	153		
8.	137	143	148	151			

pipes and conduits having quite smooth interior surfaces, and no sharp bends.* The discharge per second then is

$$q = av = c \cdot \tfrac{1}{2}a\sqrt{ds},$$

in which a is either the area of the circular cross-section or one half that section, as the case may be.

To use this table a tentative method must be employed,

* Hydraulics, p. 271.

since c depends upon the velocity of flow. For this purpose there may be taken roughly,

$$\text{mean } c = 125,$$

and then v may be computed for the given diameter and slope; a new value of c is then taken from the table and a new v computed; and thus, after two or three trials, the probable mean velocity of flow is obtained. The value of d must be expressed in feet.

For example, let it be required to find the velocity and discharge of a semicircular conduit of 6 feet diameter when laid on a grade of 0.1 feet in 100 feet. First,

$$v = 125 \times \tfrac{1}{2} \sqrt{6 \times 0.001} = 4.8 \text{ feet.}$$

For this velocity the table gives 147 for c; hence

$$v = 147 \times \tfrac{1}{2} \sqrt{0.006} = 5.7 \text{ feet.}$$

Again, from the table $c = 150$, and

$$v = 150 \times \tfrac{1}{2} \sqrt{0.006} = 5.8 \text{ feet.}$$

This shows that 150 is a little too large; for $c = 149.5$, v is found to be 5.79 feet per second, which is the final result. The discharge per second now is

$$q = 0.7854 \times \tfrac{1}{2} \times 36 \times 5.79 = 81.9 \text{ cubic feet,}$$

which is the probable flow under the given conditions.

To find the diameter of a circular conduit to discharge a given quantity under a given slope, the area a is to be expressed in terms of d in the above equation, which is then to be solved for d; thus, for a conduit which runs full,

$$d = \left(\frac{8q}{\pi c \sqrt{s}}\right)^{\frac{2}{5}},$$

and for one which is half full

$$d = \left(\frac{16q}{\pi c \sqrt{s}}\right)^{\frac{2}{5}}.$$

Here c at first may be taken as 125; then d is computed, and the approximate velocity of flow is

$$v = \frac{q}{0.7854d^2},$$

by which a value of c is selected from the table, and the computation is then repeated until the corresponding values of c and v are found to closely agree.

As an example of the determination of diameter let it be required to find d when $q = 81.9$ cubic feet per second, $s = 0.001$, and the conduit runs full. For $c = 125$ the formula gives $d = 4.9$ feet, whence $v = 4.37$ feet per second. From the table c may be now taken as 142, and repeating the computation $d = 4.64$ feet, whence $v = 4.84$ feet per second, which requires no further change in the value of c. As the tabular coefficients are based upon quite smooth interior surfaces, such as occur only in new, clean iron pipes, or with fine cement finish, it might be well to build the conduit 5 feet or 60 inches in diameter. It is seen from the previous example that a semicircular conduit of 6 feet diameter carries the same amount of water as is here provided for.

A circular conduit running full of water is a long pipe, and all the formulas and methods of Arts. 80 and 81 can be applied also to their discussion. By comparing the formulas of velocity for pipes and conduits,

$$v = \sqrt{\frac{2gdh}{fl}}, \qquad v = c \cdot \tfrac{1}{2}\sqrt{ds},$$

it is seen that

$$c = 2\sqrt{\frac{2g}{f}},$$

in which f is to be taken from Table XVI. Values of c computed in this manner will not generally agree closely with the coefficients of SMITH, partly because the values of f are given

only to three decimal places, and partly because Table XVI was constructed by regarding other discussions. An agreement within 5 per cent in mean velocities deduced by different methods is all that can generally be expected in conduit computations, and if the actual discharge agrees as closely as this with the computed discharge, the designer can be considered as a fortunate man.

All of the laws deduced in the last chapter regarding the relation between diameter and discharge, relative discharging capacity, etc., hence apply equally well to circular conduits which run either full or half full. And if the conduit be full it matters not whether it be laid truly to grade or whether a portion of it be under pressure, since in either case the slope s is the total fall h divided by the total length. Usually, however, the word conduit implies a uniform slope for considerable distances, and in this case the hydraulic gradient coincides with the surface of the flowing water.

Prob. 119. Find the discharge of a conduit when running full, its diameter being 6 feet and its fall 9.54 feet in one mile.

Prob. 120. Find the diameter of a conduit to deliver when running full 16 500 000 gallons per day, its slope being 0.00016.

ARTICLE 96. CIRCULAR CONDUITS, PARTLY FULL.

Let a circular conduit with the slope s be partly full of water, its cross-section being a and hydraulic radius r. Then the mean velocity of flow is

$$v = c\sqrt{rs},$$

and the discharge per second is

$$q = av = c.\,a\sqrt{rs}.$$

The mean velocity is hence proportional to $\sqrt{r}$ and the discharge to $a\sqrt{r}$, provided that c be a constant. Since, however, c varies slightly with r, this law of proportionality is approximate.

When a circular conduit of diameter d runs either full or half full its hydraulic radius is $\frac{1}{4}d$ (Art. 93). If it is filled to the depth d', the wetted perimeter is

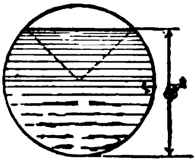


FIG. 67.

$$p = \tfrac{1}{2}\pi d + d \text{ arc sin } \frac{2d' - d}{d},$$

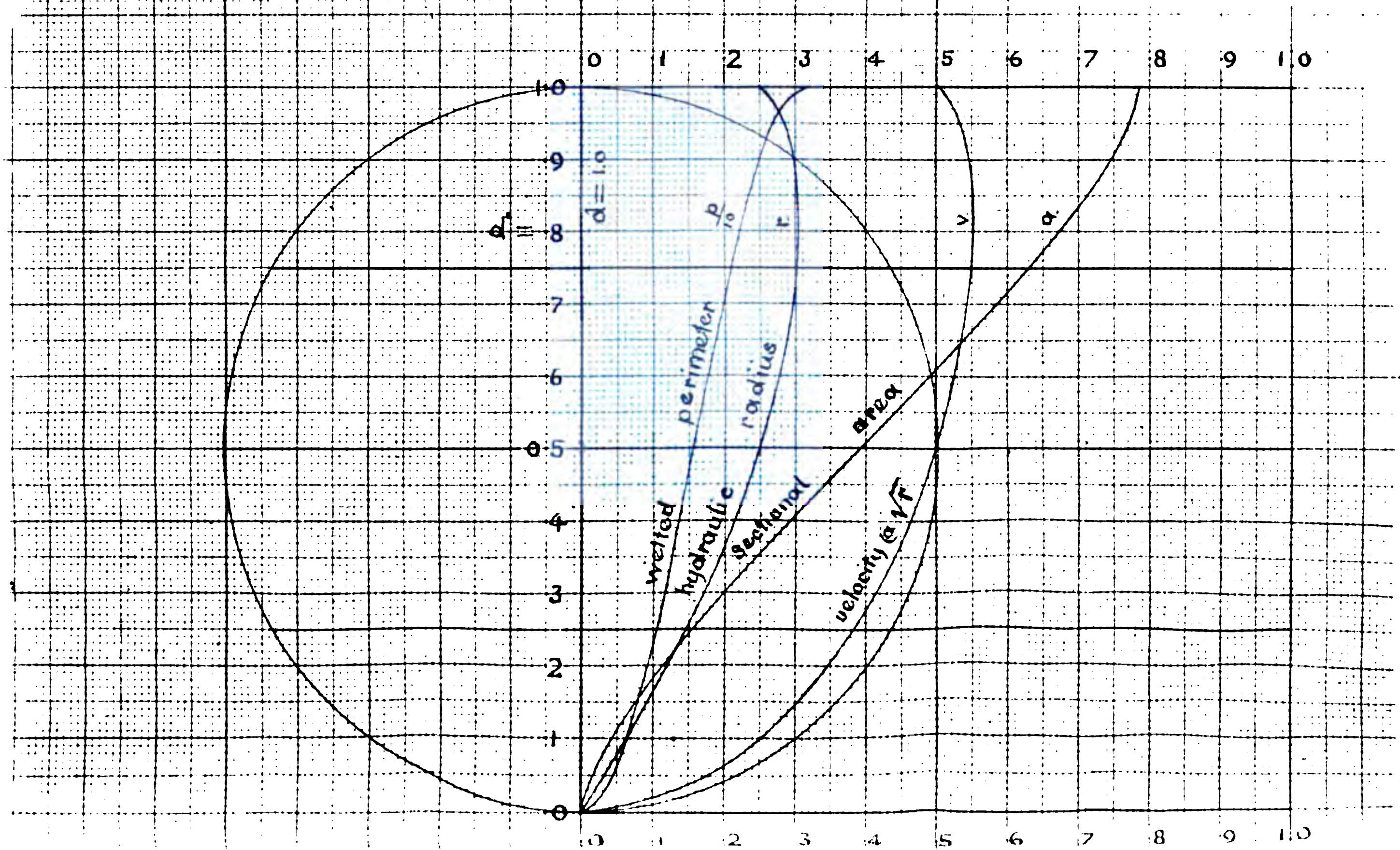
and the sectional area of the water surface is

$$a = \tfrac{1}{4}dp + (d' - \tfrac{1}{2}d)\sqrt{d'(d - d')}.$$

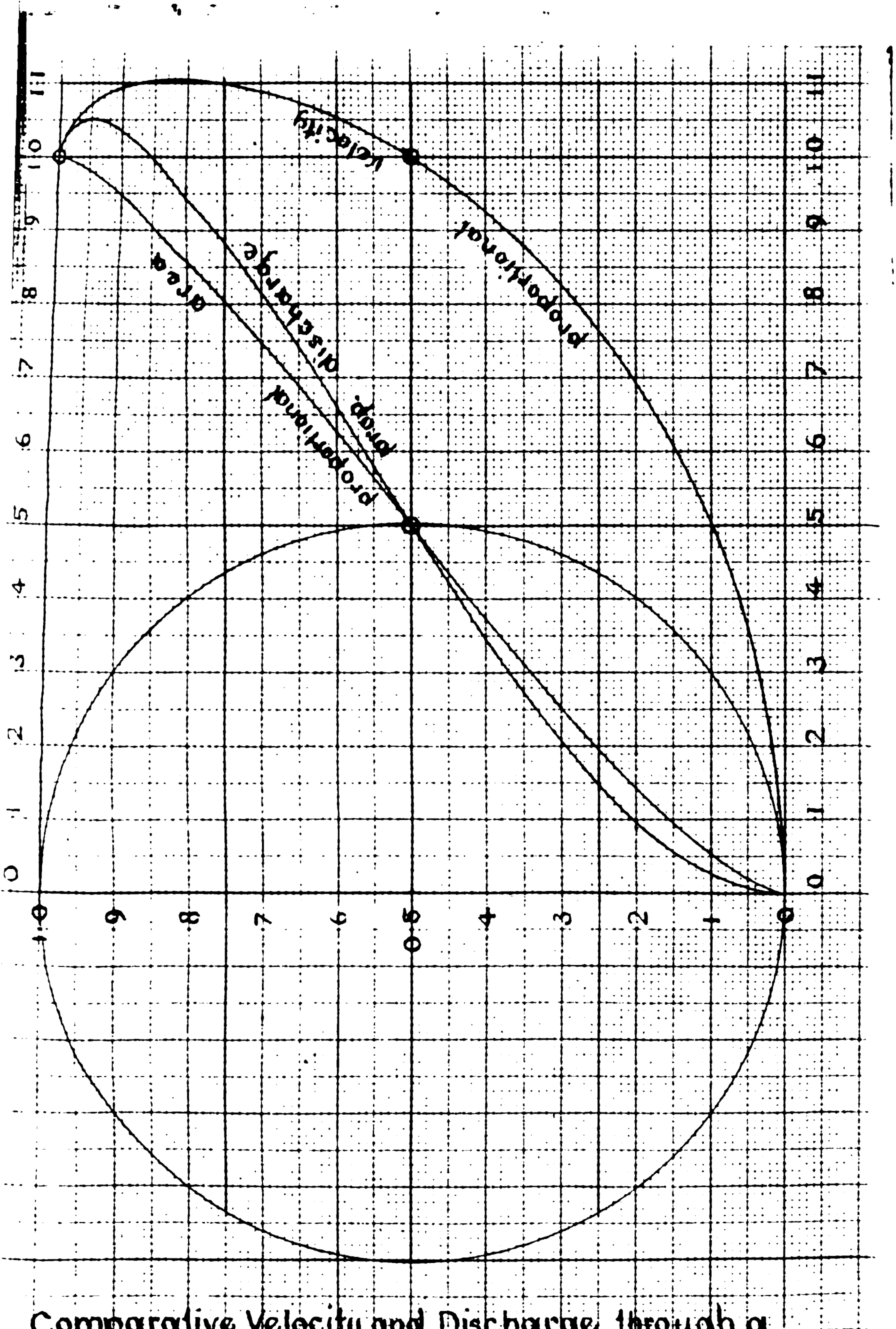
From these p and a can be computed, and then r is found by dividing a by p. The following table gives values of p, a, and r for a circle whose diameter is unity for different depths of water. To find from it the hydraulic radius for any other circle it is only necessary to multiply the tabular values of r by the given diameter d. The table shows that the greatest value of the hydraulic radius occurs when $d' = 0.81d$, and that it is but little less when $d' = 0.8d$.

TABLE XVIII. CROSS-SECTIONS OF CIRCULAR CONDUITS.

Depth d'		Wetted Perimeter p	Sectional Area a	Hydraulic Radius r	Velocity $\sqrt{r}$	Discharge $a\sqrt{r}$
Full	1.0	3.142	0.7854	0.25	0.5	0.393
	0.95	2.691	0.7708	0.2865	0.535	.413
	0.9	2.498	0.7445	0.2980	0.546	.406
	0.81	2.240	0.6815	0.3043	0.552	.376
	0.8	2.214	0.6735	0.3042	0.552	.372
	0.75	2.094	0.63184	3017	549	
	0.7	1.98324	0.58726	0.2962	0.544	.320
	0.6	1.772	0.4920	0.2776	0.527	.259
Half Full	0.5	1.571	0.3927	0.25	0.5	.196
	0.4	1.369	0.2934	0.2142	0.463	.136
	0.3	1.1592	0.1981	0.17109	0.414	.0820
	0.25	1.047	15354	[illegible]	[illegible]	
	0.2	0.927	0.1118	0.1206	0.343	.0389
	0.1	0.643	0.040[illegible]	0.0635	0.252	.0103
	0.05	0.451		[illegible]	150	
Empty	0.0	0.0	0.0	0.0	0.0	0.0



d = 1.0
d =
wetted perimeter
p/10
hydraulic radius
r
sectional area
a
v
velocity @ √r



Comparative Velocity and Discharge through a Circular Pipe of Unit Diameter at Diff. Depths of Flow. Coef. c. of v+Q. at diff. depths undetermined

May 18 1893

E.L.Cooley

2

h

F

d

r

w

c

t

o

b

In the fifth and sixth columns of the table are given values of $\sqrt{r}$ and $a\sqrt{r}$ for different depths in the circle whose diameter is unity; these are approximately proportional to the velocity and discharge which occur at those depths in a circle of any size. The table shows that the greatest velocity occurs when the depth of the water is about eight-tenths of the diameter, and that the greatest discharge occurs when the depth is about $0.95d$, or $\frac{19}{20}$ths of the diameter.

By the help of the above table the velocity and discharge may be computed when c is known, but it is not possible on account of the lack of experimental knowledge to state precise values of c for different values of r in circles of different sizes. However, it is known that an increase in r increases c, and that a decrease in r decreases c. The following experiments of DARCY and BAZIN show the extent of this variation for a semicircular conduit of 4.1 feet diameter, and they also teach that the nature of the interior surface greatly influences the values of c. Two conduits were built each with a slope $s = 0.0015$ and $d = 4.1$ feet. One was lined with neat cement, and the other with a mortar made of cement with one-third fine sand. The flow was allowed to occur with different depths, and the discharges per second were gauged by means of orifices; this enabled the velocities to be computed, and from these the values of c were found. The following are a portion of the results obtained, d' denoting the depth of water in the conduit, and all dimensions being in feet: *

For cement lining				For mortar lining			
d'	r	v	c	d'	r	v	c
2.05	1.029	6.06	154	2.04	1.022	5.55	142
1.83	0.949	5.75	152	1.80	0.941	5.20	138
1.61	0.867	5.29	147	1.69	0.900	4.94	135
1.34	0.750	4.87	145	1.41	0.787	4.51	131
1.03	0.605	4.16	138	1.09	0.635	3.87	125
0.83	0.503	3.72	136	0.88	0.529	3.43	122
0.59	0.366	3.02	129	0.61	0.379	2.87	120

* SMITH's Hydraulics, p. 176.

It is here seen that c decreases quite uniformly with r, and that the velocities for the mortar lining are 8 or 10 per cent less than for the neat cement lining.

The value of the coefficient c for these experiments may be roughly expressed by the formula

$$c = c' - 16(\tfrac{1}{2}d - d'),$$

in which c' is the coefficient for the conduit when running half full. How this will apply to different diameters and velocities is not known; when d' is greater than $0.8d$ it will probably prove incorrect. In practice, however, computations on the flow in partly filled conduits are of rare occurrence.

Prob. 121. Compute the hydraulic radius for a circular conduit when it is three-fourths filled with water, and also the mean velocity if it be lined with pure cement and laid on a grade of 0.15 per 100, the diameter being 4.1 feet.

ARTICLE 97. OPEN RECTANGULAR CONDUITS.

In designing an open rectangular trough or conduit to carry water there is a certain ratio of breadth to depth which is most advantageous, because that thereby either the discharge is the greatest or the least amount of material is required for its construction. This advantageous proportion is the one which offers the least frictional resistance to the flow; in a very wide and shallow trough the friction would be great, and the same would be the case in one of small width and large depth. It is now to be shown that the least friction, and hence the best proportions, results when the width is double of the depth.

The head lost in friction is directly proportional to the wetted perimeter and inversely proportional to the area of the water cross-section (Art. 94). In order that this may be the

least possible, the wetted perimeter should be a minimum for a given area, or the area should be a maximum for a given wetted perimeter. But the ratio of the area to the perimeter is the hydraulic radius

$$r = \frac{a}{p},$$

which therefore is to be a maximum, subject to the other conditions of the problem, in order to secure the most advantageous cross-section. This is an approximate general rule, applicable to all kinds of channels, and it is plain that the circle fulfils the requirement in a higher degree than any other figure.

For an open rectangular conduit of breadth b and depth d the value of the hydraulic radius is

$$r = \frac{bd}{b + 2d}.$$

If it be required to find the most advantageous section for a given wetted perimeter, this may be written

$$r = \frac{b(p - b)}{2p};$$

and this is seen to be a maximum when $b = \frac{1}{2}p$, that is, when $b = 2d$, or the breadth is double the depth. If, however, it be required to determine the most advantageous section for a given area, the value of the hydraulic radius may be written

$$r = \frac{a}{b + 2\dfrac{a}{b}};$$

and by equating the first derivative to zero, there is found $b^2 = 2a$, from which $b^2 = 2bd$, or $b = 2d$, as before.

Again, if it be required to find the most advantageous section to carry q cubic feet of water per second, the hydraulic radius

$$r = \frac{bd}{b + 2d}$$

is to be made a maximum, subject to the condition

$$q = c \cdot a \sqrt{rs} = cs^{\frac{1}{2}} \sqrt{\frac{b^3d^3}{b + 2d}}.$$

Regarding c as a constant, the values of b and d which render r a maximum can be ascertained by the rules of the higher analysis, and there is also found for this case the relation $b = 2d$, or the breadth is double the depth.

The velocity and discharge through a rectangular conduit are expressed by the general equations

$$v = c\sqrt{rs}, \qquad q = av,$$

and are computed without difficulty for any given case when the coefficient c is known. To ascertain this, however, is not easy on account of the lack of experiments by which alone its value can be ascertained. When the depth of the water in the conduit is one-half of its width, thus giving the most advantageous section, the values of c for smooth interior surfaces may be estimated from the table in Art. 96 for circular conduits, although c is probably smaller for rectangles than for circles of equal area. When the depth of the water is less or greater than $\frac{1}{2}d$, it must be remembered that c increases with r. The value of c also is subject to slight variations with the slope s, and to great variations with the degree of roughness of the surface.

The following table, derived from SMITH'S discussion of the experiments of DARCY and BAZIN, gives values of c for a num-

ber of wooden and masonry conduits with rectangular sections, all of which were laid on the grade of 0.49 feet per 100, or

TABLE XIX. COEFFICIENTS FOR RECTANGULAR CONDUITS.

Unplaned Plank. $b = 3.93$ feet.		Unplaned Plank. $b = 6.53$ feet.		Pure Cement. $b = 5.94$ feet.		Brick. $b = 6.27$ feet.	
d	c	d	c	d	c	d	c
0.27	99	0.20	89	0.18	116	0.20	89
.41	108	.30	101	.28	125	.31	98
.67	112	.46	109	.43	132	.49	104
.89	114	.60	113	.56	135	.57	105
1.00	114	.72	116	.63	136	.65	104
1.19	116	.78	116	.69	136	.71	106
1.29	117	.89	118	.80	137	.85	107
1.46	118	.94	120	.91	138	.97	110

$s = 0.0049$. The great influence of roughness of surface in diminishing the coefficient is here plainly seen. For masonry conduits with hammer-dressed surfaces c may be as low as 60 or 50, particularly when covered with moss and slime.

Prob. 122. Compare the discharge of a trough 1×3 feet with that of two troughs each 1×2 feet.

Prob. 123. Find the size of a trough, whose width is double its depth, which will deliver 125 cubic feet per minute when its slope is 0.002, taking c as 100. Ans. $d = 0.64$ feet.

ARTICLE 98. TRAPEZOIDAL SECTIONS.

Ditches and conduits are often built with a bottom nearly flat and with side slopes, thus forming a trapezoidal section. The side slope is fixed by the nature of the soil or by other circumstances, the grade is given, and it may be then required to ascertain the relation between the bottom width and the depth of water, in order that the section shall be the most advantageous. This can be done by the same reasoning as used

for the rectangle in the last article, but it may be well to employ a different method, and thus be able to consider the subject in a new light.

Let the trapezoidal channel have the bottom width b, the depth d, and let θ be the angle made by the side slopes with the horizontal. Let it be required to discharge q cubic feet per second; then

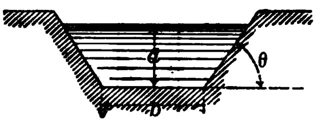


FIG. 68.

$$q = ca\sqrt{rs}.$$

Now the most advantageous proportions may be said to be those that will render the cross-section a a minimum, for thus the least excavation will be required. The above equation may be written

$$\frac{q^2}{c^2s} = \frac{a^3}{p}.$$

In this p is to be replaced by its value in terms of a and d, and then the value of d is to be found which renders a a minimum. For this purpose the figure gives

$$a = d(b + d \cot \theta);$$

$$p = b + \frac{2d}{\sin \theta} = \frac{a}{d} + d\left(\frac{2}{\sin \theta} - \cot \theta\right);$$

from which the equation becomes

$$q^2\frac{a}{d} + q^2 d\left(\frac{2}{\sin \theta} - \cot \theta\right) = c^2sa^3.$$

Obtaining the first derivative of a with respect to d, and equating it to zero, there is found

$$\left(\frac{2}{\sin \theta} - \cot \theta\right)d^2 = a;$$

and replacing for a its value, there results

$$b = d\left(\frac{2}{\sin \theta} - 2 \cot \theta\right); \quad . \quad . \quad . \quad . \quad . \quad (71)$$

which is the relation that gives the most advantageous cross-section. If $\theta = 90°$, the trapezoid becomes a rectangle, and $b = 2d$, as previously deduced. As c has been regarded as a constant in this investigation, the conclusion is not a rigorous one, although it may be safely followed in practice. It is to be expected, as in all cases of a maximum, that quite considerable variations in the ratio $b : d$ may occur without materially affecting the value of a.

When the value of c is known, the general formulas $v = c\sqrt{rs}$ and $q = av$ may be used to obtain a rough approximation to the discharge. The formula of KUTTER (Art. 101) may be used to determine c when the nature of the bed of the channel is known. In any important case, however, computations cannot be trusted to give reliable values of the discharge on account of the uncertainty regarding the coefficient, and an actual gauging of the flow should be made. This is best effected by a weir, but if that should prove too expensive, the methods explained in Chap. IX may be employed to give more precise results than can usually be determined by any computation.

The problem of determining the size of a trapezoidal channel to carry a given quantity of water, does not require c to be determined so closely. For this purpose the following values may be used, the lower ones being for small cross-sections with rough and foul surfaces, and the higher ones for quite smooth surfaces:

For unplaned plank,	$c =$ 100 to 120
For smooth masonry,	$c =$ 90 to 110
For clean earth,	$c =$ 60 to 80
For stony earth,	$c =$ 40 to 60
For rough stone,	$c =$ 35 to 50
For earth foul with weeds,	$c =$ 30 to 50

To solve this problem, let a and p be replaced by their values in terms of b and d. The discharge then is

$$q = cd(b + d \cot \theta)\sqrt{\frac{d(b + d \cot \theta)s \sin \theta}{b \sin \theta + 2d}}.$$

Now when q, c, θ, and s are known, the equation contains two unknown quantities, b and d. If the section is to be the most advantageous, b can be replaced by its value in terms of d as above found, and the equation then has but one unknown. Or in general, if $b = md$, where m is any assumed number, the solution gives

$$d^5 = \frac{q^2(m \sin \theta + 2)}{c^2 s(m + \cot \theta)^3 \sin \theta}.$$

For the particular case where the side slopes are 1 to 1 or $\theta = 45°$, and the bottom width is to be equal to the water depth, or $m = 1$, this becomes

$$d = 0.863\left(\frac{q^2}{c^2 s}\right)^{\frac{1}{5}}.$$

These formulas are analogous to those for finding the diameter of pipes and circular conduits, and the numerical operations are in all respects similar. It is plain that by assigning different values to m numerous sections may be determined which will satisfy the imposed conditions, and usually the one is to be selected that will give both a safe velocity and a minimum cost. In Art. 103 will be found an example of the determination of the size of a trapezoidal canal.

Prob. 124. If the value of c is 71, compute the depth of a trapezoidal section to carry 200 cubic feet of water per second, θ being 45°, the slope s being 0.001, and the bottom width being equal to the depth. Compute also the mean velocity for the section.

ARTICLE 99. HORSE-SHOE CONDUITS.

In Fig. 69 is given an outline cross-section of the Sudbury conduit, the flow of which was gauged by FTELEY and STEARNS, whose discussions have determined a formula for its mean velocity. The section consists of a part of a circle of 9.0 feet diameter, having an invert of 13.22 feet radius, whose span is 8.3 feet and depression 0.7 feet, the axial depth of the conduit being 7.7 feet. The conduit is lined with brick, having cement joints one quarter of an inch thick. The flow was allowed to occur with different depths, for each of which the discharge was determined by weir measurement. A discussion of the results led to the conclusion that in the portion with the brick lining the coefficient c had the value $127r^{0.12}$ when r is in feet, and hence

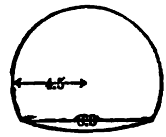


FIG. 69.

$$v = 127r^{0.12}\sqrt{rs} = 127r^{0.62}s^{0.5}. \quad . \quad . \quad . \quad . \quad (72)$$

In a portion of the conduit where the brick lining was coated with pure cement the coefficient was found to be from 7 to 8 per cent greater than 127. In another portion where the brick lining was covered with a cement wash laid on with a brush the coefficient was from 1 to 3 per cent greater. For a long tunnel in which the rock sides were ragged, but with a smooth cement floor, it was found to be about 40 per cent less.*

These results clearly show that the coefficient c increases with r, and that it is greatly influenced by the nature of the interior surfaces. For sections of smaller area than that above given the value of c is undoubtedly less than $127r^{0.12}$, and for those of larger area it is greater; the extent of variation may perhaps be inferred from the table in Art. 95. The general

* Transactions American Society Civil Engineers 1883, vol. xii. p. 114.

slope of the Sudbury conduit is about one foot per mile, and c is also subject to variation with s, as well as with the temperature of the water. Although the above formula is a special one, applicable to a single conduit, it is nevertheless of great value, as it presents the only existing evidence regarding the coefficients for large aqueducts.

Prob. 125. The actual discharge of the Sudbury conduit is about 60 080 000 gallons per 24 hours when the water is 4 feet deep, a being 33.31 sq. feet, $p = 15.21$ feet, and $s = 0.0001895$. Compute the discharge by the use of the above formula.

ARTICLE 100. LAMPE'S FORMULA.

The formula given in Art. 91 for the mean velocity of flow in long circular pipes can be also applied to conduits with very smooth surfaces. Replacing for the ratio $h \div l$ the slope s, and for d its equivalent $4r$, it becomes

$$v = 203 r^{0.694} s^{0.555}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

This formula may be also written

$$v = 203 r^{0.194} s^{0.055} \sqrt{rs}, \quad . \quad . \quad . \quad . \quad . \quad (73)'$$

in which the quantity preceding the radical in the second member is the coefficient c. According to this empirical expression c increases both with r and s, but only slightly with the latter. It is probable that this formula represents quite accurately the laws of flow in conduits, but the varying degree of roughness of surface is not taken into account by it, so that in general it can only be used to furnish approximate results, except for the case of metal pipes or similar smooth surfaces. For this purpose the formulas for q and d, given in Art. 91, may be directly used for circular sections. It is probable that future researches may show that a formula similar to the above may fairly repre-

sent all cases, the constant 203 being varied with the roughness of the surface.

Prob. 126. Solve Prob. 125 by the use of LAMPE'S formula, and compare the error of the result with that as deduced by the special formula for the conduit.

ARTICLE 101. KUTTER'S FORMULA.

The researches of GANGUILLET and KUTTER have furnished a general expression for the coefficient c in the formula for mean velocity,

$$v = c\sqrt{rs},$$

by which its value can be computed for any given case when the nature of the interior surface is known. This expression is, for English measures,

$$c = \frac{\dfrac{1.811}{\alpha} + 41.65 + \dfrac{0.00281}{s}}{1 + \dfrac{\alpha}{\sqrt{r}}\left(41.65 + \dfrac{0.00281}{s}\right)}, \quad . \quad . \quad . \quad . \quad (74)$$

in which α is an abstract number whose value depends only upon the roughness of the surface, and

$\alpha = 0.009$ for well-planed timber;

$\alpha = 0.010$ for neat cement;

$\alpha = 0.011$ for cement with one-third sand;

$\alpha = 0.012$ for unplaned timber;

$\alpha = 0.013$ for ashlar and brickwork;

$\alpha = 0.015$ for unclean surfaces in sewers and conduits;

$\alpha = 0.017$ for rubble masonry;

$\alpha = 0.020$ for canals in very firm gravel;

$\alpha = 0.025$ for canals and rivers free from stones and weeds;

$\alpha = 0.030$ for canals and rivers with some stones and weeds;

$\alpha = 0.035$ for canals and rivers in bad order.

By inserting this value of c in the formula for v, the mean velocity is made to depend upon r, s, and the roughness of the surface.

The formula of KUTTER has received a wide acceptance on account of its application to all kinds of surfaces. Notwithstanding that it is purely empirical, and hence not perfect, it is to be regarded as a formula of great value, so that no design for a conduit or channel should be completed without employing it in the investigation, even if the final construction be not based upon it. In sewer work it is extensively employed, α being taken as about 0.015. The formula shows that c always increases with r, that it decreases with s when r is greater than 3.28 feet, and that it increases with s when r is less than 3.28 feet. When r equals 3.28 feet the value of c is simply $\frac{1.811}{\alpha}$. It is not likely that future investigations will confirm these laws of variation in all respects.

In the following articles are given values of c for a few cases, and these might be greatly extended, as has been done by KUTTER and others. But this is scarcely necessary except for special lines of investigation, since for single cases there is no difficulty in directly computing it for given data. For instance, take a rectangular trough of unplaned plank 3.93 feet wide on a slope of 0.0049, the water being 1.29 feet deep. Here

$$s = 0.0049$$

and

$$r = \frac{3.93 \times 1.29}{3.93 + 2.58} = 0.779 \text{ feet.}$$

Then α being 0.012, the value of c is found to be

$$c = \frac{\frac{1.811}{0.012} + 41.65 + \frac{0.00281}{0.0049}}{1 + \frac{0.012}{\sqrt{0.779}}\left(41.65 + \frac{0.00281}{0.0049}\right)} = 123.$$

The data here used are taken from the table in Art. 97, where the actual value of c is given as 117; hence in this case KUTTER'S formula is about 5 per cent in excess. As a second example, the following data from the same table will be taken: a rectangular conduit in pure cement, $b = 5.94$ feet, $d = 0.91$ feet, $s = 0.0049$. Here $\alpha = 0.010$, and $r = 0.697$ feet. Inserting all values in the formula, there is found $c = 148$, which is 8 per cent greater than the true value, 138. Thus is shown the fact that errors of 5 and 10 per cent are to be regarded as common in calculations on the flow of water in conduits and canals.

Prob. 127. Compute by KUTTER'S formula the discharge for the data in Prob. 125.

ARTICLE 102. SEWERS.

Sewers smaller in diameter than 18 inches are always circular in section. When larger than this they are built with the section either circular, egg-shaped, or of the horseshoe form. The last shape is very disadvantageous when a small quantity of sewage is flowing, for the wetted perimeter is then large compared with the area, the hydraulic radius is small, and the velocity becomes low, so that a deposit of the foul materials results. As the slope of sewer lines is often very slight, it is important that such a form of cross-section should be adopted to render the velocity of flow sufficient to prevent this deposit. A velocity of 2 feet per second is found to be about the minimum allowable limit, and 4 feet per second need not be usually exceeded.

The egg-shaped section is designed so that the hydraulic radius may not become small even when a small amount of sewage is flowing. One of the most common forms is that shown in Fig. 70, where the greatest width DD is two-thirds of the depth HM. The arch DHD is a semicircle described from A as a centre. The invert LML is a portion of a circle described from B as a centre, the distance BA being three-fourths of DD and the radius BM being one-half of AD. Each side DL is described from a centre C so as to be tangent to the arch and invert. These relations may be expressed more concisely by

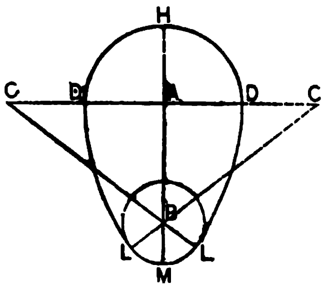


FIG. 70.

$$HM = 1\tfrac{1}{2}D, \quad AB = \tfrac{3}{4}D, \quad BM = \tfrac{1}{4}D, \quad CL = 1\tfrac{1}{2}D,$$

in which D is the horizontal diameter DD.

Computations on egg-shaped sewers are usually confined to three cases, namely, when flowing full, two-thirds full, and one-third full. The values of the sectional areas, wetted perimeters, and hydraulic radii for these cases, as given by FLYNN,* are

	a	p	r
Full	$1.1485D^2$	$3.965D$	$0.2897D$
Two-thirds full	$0.7558D^2$	$2.394D$	$0.3157D$
One-third full	$0.2840D^2$	$1.375D$	$0.2066D$

This shows that the hydraulic radius, and hence the velocity, is but little less when flowing one-third full than when flowing with full section.

Egg-shaped sewers and small circular ones are formed by laying consecutive lengths of clay or cement pipe whose interior

* Van Nostrand's Magazine, 1883, vol. xxviii. p. 138.

surfaces are quite smooth when new, but may become foul after use. Large sewers of circular section are made of brick, and are more apt to become foul than smaller ones. In the separate system, where systematic flushing is employed and the pipes are small, foulness of surface is not so common as in the combined system, where the storm water is alone used for this purpose. In the latter case the sizes are computed for the volume of storm water to be discharged, the amount of sewage being very small in comparison.

The discharge of a sewer pipe enters it at intervals along its length, and hence the flow is not uniform. The depth of the flow increases along the length, and at junctions the size of the pipe is enlarged. The strict investigation of the problem of flow is accordingly one of great complexity. But considering the fact that the sewer is rarely filled, and that it should be made large enough to provide for contingencies and future extensions, it appears that great precision is unnecessary. The universal practice, therefore, is to discuss a sewer for the condition of maximum discharge, regarding it as a channel with uniform flow. The main problem is that of the determination of size; if the form be circular, the diameter is found, as in Art. 95, by

$$d = \left(\frac{8q}{\pi c \sqrt{s}}\right)^{\frac{2}{5}} = 1.45 \left(\frac{q}{c \sqrt{s}}\right)^{\frac{2}{5}}.$$

If the form be egg-shaped and of the proportions above explained, the discharge when running full is

$$q = ac \sqrt{rs} = 1.1485 D^2 c \sqrt{0.2897 Ds},$$

from which the value of D is found to be

$$D = 1.21 \left(\frac{q}{c \sqrt{s}}\right)^{\frac{2}{5}}.$$

Thus when q has been determined and c is known the required sizes for given slopes can be computed. The velocity should

also be found in order to ascertain if it be high enough to prevent deposit (Art. 108).

Few or no experiments exist from which to directly determine the coefficient c for the flow in sewers, but since the sewage is mostly water, it may be approximately ascertained from the values for similar surfaces. KUTTER'S formula has been extensively employed for this purpose, using 0.015 for the coefficient of roughness. The following table gives values of c for three different slopes and for two classes of surfaces. The values for the degree of roughness represented by $\alpha = 0.017$

TABLE XX. COEFFICIENTS FOR SEWERS.

Hydraulic Radius r in Feet.	$s = 0.00005$		$s = 0.0001$		$s = 0.01$	
	$a = 0.015$	$a = 0.017$	$a = 0.015$	$a = 0.017$	$a = 0.015$	$a = 0.017$
0.2	52	43	58	48	68	57
0.3	60	51	66	56	76	64
0.4	65	56	73	61	83	70
0.6	76	65	82	70	90	76
0.8	82	72	87	76	95	82
1.	88	77	92	80	99	87
1.5	100	86	103	89	108	93
2.	106	94	108	96	111	99
3.	116	103	118	104	118	105

are applicable to sewers with quite rough surfaces of masonry; those for $\alpha = 0.015$ are applicable to sewers with ordinary smooth surfaces, somewhat fouled or tuberculated by deposits, and are the ones to be generally used in computations. By the help of this table and the general equations for mean velocity and discharge all problems relating to flow in sewers can be readily solved.

Prob. 128. The grade of a sewer is one foot in 960, and its discharge is to be 65 cubic feet per second. What is the diameter of the sewer if circular? Ans. $d = 4.8$ feet.

ARTICLE 103. DITCHES AND CANALS.

Ditches for irrigating purposes are of a trapezoidal section, and the slope is determined by the fall between the point from which the water is taken and the place of delivery. If the fall is large it may not be possible to construct the ditch in a straight line between the two points, even if the topography of the country should permit, on account of the high velocity which would result. A velocity exceeding 2 feet per second may often prove injurious in wearing the bed of the channel unless protected by riprap or other lining. For this reason as well as for others the alignment of ditches and canals is often circuitous.

The principles of the preceding articles are sufficient to solve all usual problems of uniform flow in such channels when the values of c are known. These are perhaps best determined by KUTTER'S formula, and for greater convenience a table is

TABLE XXI. COEFFICIENTS FOR CHANNELS IN EARTH.

Hydraulic Radius r in Feet.	$s = 0.00005$		$s = 0.0001$		$s = 0.01$	
	$n = 0.025$	$n = 0.030$	$n = 0.025$	$n = 0.030$	$n = 0.025$	$n = 0.030$
0.5	38	31	41	33	47	37
1.	49	40	52	42	56	45
1.5	57	47	59	48	62	51
2.	64	52	65	53	67	54
3.	72	59	72	59	72	60
4.	77	64	77	64	76	63
5.	81	68	80	68	79	66
6.	86	72	84	71	80	68
8.	91	76	87	74	82	70
10.	96	80	91	80	85	73
15.	105	89	97	84	90	77
25.	114	100	101	92	95	82

here given, showing their average values for three slopes and two degrees of roughness.

As an example of the use of the table let it be required to find the width and depth of a ditch of most advantageous cross-section, whose channel is to be in tolerably good order, so that $\alpha = 0.025$. The amount of water to be delivered is 200 cubic feet per second and the grade is 1 in 1000, the side slopes of the channel being 1 to 1. From Art. 98 the relation between the bottom width and the depth of the water is, since $\theta = 45°$,

$$b = d\left(\frac{2}{\sin\theta} - 2\cot\theta\right) = 0.828d.$$

The area of the cross-section then is

$$a = d(b + d\cot\theta) = 1.828d^2,$$

and the wetted perimeter is

$$p = b + \frac{2d}{\sin\theta} = 3.656d,$$

whence the hydraulic radius is found to be

$$r = \frac{1.828d^2}{3.656d} = 0.5d.$$

It is indeed a general rule, which might properly have been set forth in Art. 98, that the hydraulic radius is one-half the depth of the water in trapezoidal channels of most advantageous cross-section. Now, since d is unknown c cannot be taken from the table, and as a first approximation let it be supposed to be 60. Then in the general formula for discharge the above values are substituted, giving

$$200 = 60 \times 1.828d^2 \sqrt{0.5d \times 0.001},$$

from which d is found to be 5.8 feet. Accordingly $r = 2.9$ feet, and from the table c is about 71. Repeating the computation with this value of c there is found $d = 5.44$ feet, which,

considering the uncertainty of c, is sufficiently close. The depth may then be made 5.5 feet, and the bottom width will be

$$b = 0.828 \times 5.5 = 4.55 \text{ feet,}$$

and the sectional area is

$$a = 1.828 \times 5.5^2 = 55.3 \text{ square feet,}$$

which gives for the velocity

$$v = \frac{200}{55.3} = 3.62 \text{ feet per second.}$$

This completes the investigation if the velocity is regarded as satisfactory. But for most earths this would be too high, and accordingly the section must be made wider and of less depth in order to reduce the hydraulic radius and diminish the velocity.

The following statements show approximately the velocities which are required to move different materials:

0.25 feet per second moves fine clay,
0.5 feet per second moves loam and earth,
1.0 feet per second moves sand,
2.0 feet per second moves gravel,
3.0 feet per second moves pebbles 1 inch in size,
4.0 feet per second moves spalls and stones,
6.0 feet per second moves large stones.

The mean velocity in a channel may be somewhat larger than these values before the materials will move, because the velocities along the wetted perimeter are smaller than the mean velocity. More will be found on this subject in Art. 107. f 251

Prob. 129. Compute the mean velocity in a ditch which is to discharge 200 cubic feet per second on a grade of 1 in 1000 when its bottom width is 16 feet and the side slopes are 1 to 1.

Ans. $d = 3.09$ feet, $v = 3.4$ feet, per second.

ART. 104. LOSSES OF HEAD.

The only loss of head thus far considered is that due to friction, but other sources of loss may often exist. As in the flow in pipes, these may be classified as losses at entrance, losses due to curvature, and losses caused by obstructions in the channel or by changes in the area of cross-section.

When water is admitted to a channel from a reservoir or pond through a rectangular sluice there occurs a contraction similar to that at the entrance into a pipe, and which may be often observed in a slight depression of the surface, as at D in the diagram. At this point, therefore, the velocity is greater than the mean velocity v, and a loss of energy or head results from the subsequent expansion, which is approximately measured by the difference of the depths d_1 and d_2, the former being taken at the entrance of the channel, and the latter below the depression where the uniform flow is fully established. According to the experiments of DUBUAT, the loss of head is measured by

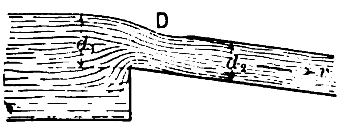


FIG. 71.

$$d_1 - d_2 = m\frac{v^2}{2g},$$

in which m ranges between 0 and 2 according to the condition of the entrance. If the channel be small compared with the reservoir, and both the bottom and side edges of the entrance be square, m may be nearly 2; but if these edges be rounded, m may be very small, particularly if the bottom contraction is suppressed. All the remarks in Chapter IV relating to suppression of the contraction apply here, and in a short channel or flume it may be important to prevent this loss of head by a rounded or curved approach.

The loss of head due to bends or curves in the channel is small if the curvature be slight. Undoubtedly every curve offers a resistance to the change in direction of the velocity, and thus requires an additional head to cause the flow beyond that needed to overcome the frictional resistances. Several formulas have been proposed to express this loss, but they all seem unsatisfactory, and hence will not be presented here, particularly as the data for determining their constants are very scant. It will be plain that the loss of head due to a curve increases with its length and decreases with its radius. Art. 131 gives a discussion concerning the cause of losses in bends and curves.

The losses of head caused by sudden enlargement or by sudden contraction of the cross-section of a channel may be estimated by the rules deduced in Arts. 68 and 69. In order to avoid these losses changes of section should be made gradually, so that energy may not be lost in impact. Obstructions or submerged dams may be regarded as causing sudden changes of section, and the accompanying losses of head are governed by similar laws. The numerical estimation of these losses will generally be difficult, but the principles which control them will often prove useful in arranging the design of a channel so that the maximum work of the water can be rendered available. But as all losses of head are directly proportional to the velocity-head $\frac{v^2}{2g}$, it is plain that they can be rendered inappreciable by giving to the channel such dimensions as will render the mean velocity very small. This may sometimes be important in a short conduit or flume which conveys water from a pond or reservoir to a hydraulic motor, particularly in cases where the supply is scant, and where all the available head is required to be utilized.

Prob. 130. Explain what will happen when in a channel

which conveys 50 cubic feet of water per second the cross-section suddenly changes from 5 to 25 square feet.

ARTICLE 105. THE ENERGY OF THE FLOW.

If all the filaments of a stream of water flowing in a pipe, conduit, or canal have the same uniform velocity v, the potential energy per second is the weight W of the discharge per second multiplied by its velocity-head $\frac{v^2}{2g}$; or if a be the cross-section of the stream and w the weight of a cubic unit of water, the energy is

$$K = W\frac{v^2}{2g} = wq\frac{v^2}{2g} = wa\frac{v^3}{2g},$$

in which W is the total weight of water delivered per second and w is the weight of one cubic foot. In this case, then, the work of a stream of constant cross-section varies as the cube of its velocity.

The velocities of the filaments in a cross-section are, however, not uniform, some being less and others greater than the mean velocity v, so that the above expression for K does not truly represent the energy of the discharge. Let the cross-section be divided into a number n of elementary sections, each of which is equal to a'; then the mean velocity is

$$v = \frac{a'}{a}(v_1 + v_2 + \ldots + v_n) = \frac{v_1 + v_2 + \ldots + v_n}{n},$$

and the true energy of the discharge is

$$K' = wa'\left(\frac{v_1^3}{2g} + \frac{v_2^3}{2g} + \ldots + \frac{v_n^3}{2g}\right).$$

It is now to be shown that K' is always greater than K, except for the particular case when the velocities of the individual filaments are all equal.

For this purpose let the equation for K' be divided by that for K, and n be placed for the ratio $a \div a'$, giving

$$\frac{K'}{K} = \frac{v_1^3 + v_2^3 + \ldots + v_n^3}{nv^3} = \frac{\Sigma v_1^3}{nv^3}.$$

Now let u be the difference between the mean v and any individual velocity, so that $v_1 = v \pm u_1$, $v_2 = v \pm u_2$, etc.; then

$$\Sigma v_1^3 = \Sigma v^3 \pm 3\Sigma v^2 u + 3\Sigma vu^2 \pm \Sigma u^3.$$

But it is a property of the arithmetical mean that $\Sigma u = 0$; hence the terms containing u and u^3 disappear, and since $nv^3 = \Sigma v^3$, the expression becomes

$$\frac{K'}{K} = 1 + \frac{3\Sigma u^2}{nv^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Therefore the true energy K' is always greater than that represented by K.

It is difficult, if not impossible, to give even a general statement of the percentage which is to be added to the energy K in order to find the true energy K'. In a circular pipe or small

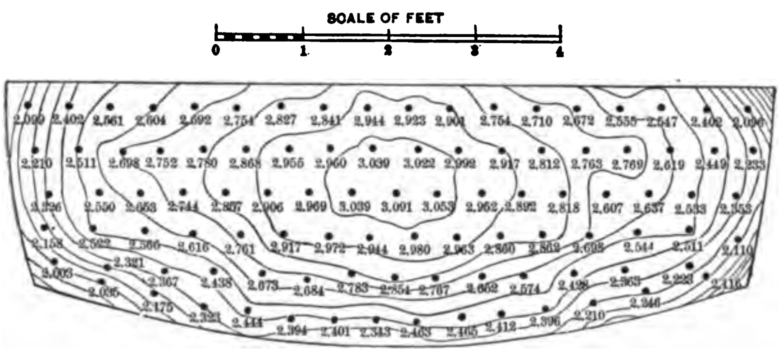


FIG. 72.

trough the velocities may not greatly differ, so that K and K' may closely agree. The experiments of FTELEY and STEARNS

on the Sudbury conduit furnish the means of computing this percentage for several cases, one of which is represented in Fig. 72.* This shows the cross-section of the conduit when the water was about 3 feet deep, the dots being the points at which the velocities were measured by a current meter (Art. 109), and the figures giving the observed values in feet per second. The number of these velocities is $n = 97$, and their mean is $v = 2.620$ feet per second. Subtracting this from each individual velocity there are found 97 values of u, the sum of whose squares is $\Sigma u^2 = 7.342$. These values inserted in the formula (75) give

$$K' = K\left(1 + 3\,\frac{7.342}{97(2.62)^2}\right) = 1.033K;$$

and therefore in this case the true energy is 3.3 per cent greater than that determined by using the mean velocity v.

Prob. 131. The discharge of the Sudbury conduit under the conditions above described was 64.43 cubic feet per second. Compute the theoretic horse-power of the flow.

* Transactions American Society of Civil Engineers, 1883, vol. xii. p. 324.

CHAPTER IX.

FLOW IN RIVERS.

ARTICLE 106. BROOKS AND RIVERS.

No branch of Hydraulics has received more detailed investigation than that of the flow in river channels, and yet the subject is but imperfectly understood. The great object of all these investigations has been to devise a simple method of determining the mean velocity and discharge without the necessity of expensive field operations. In general it may be said that this end has not yet been attained, even for the case of uniform flow. Of the various formulas proposed to represent the relation of mean velocity to the hydraulic radius and the slope, none have proved to be of general practical value except the empirical expression used in the last chapter, and this is often inapplicable on account of the difficulty of measuring s and determining c. The fundamental equations for discussing the laws of variation in the mean velocity v and in the discharge q are

$$v = c\sqrt{rs}, \qquad q = a\,.\,c\sqrt{rs};$$

and all the general principles of the last chapter are to be taken as directly applicable to uniform flow in natural channels.

KUTTER'S formula for the value of c is probably the best in the present state of science, although it is now generally recognized that it gives too large values for small slopes. In using it the coefficients for rivers in good condition may be taken from Art. 103, but for bad regimen α is to be taken at 0.03, and for wild torrents at 0.04 or 0.05. It is, however, too much to

expect that a single formula should accurately express the mean velocity in small brooks and large rivers, and the general opinion now is that efforts to establish such an expression will not prove successful. In the present state of the science no engineer can afford in any case of importance to rely upon a formula to furnish anything more than a rough approximation to the discharge in river channels, but actual field measurements of velocity must be made.

When the above formulas are used to determine the discharge of a river a long straight portion or reach should be selected, where the cross-sections are uniform in shape and size. The width of the stream is then divided into a number of parts, and soundings taken at each point of division. The data are thus obtained for computing the area a and the wetted perimeter p, from which the hydraulic depth r is derived. To determine the slope s a length l is to be measured, at each end of which bench marks are established whose difference of elevation is found by precise levels. The elevations of the water surfaces below these benches are then to be simultaneously taken, whence the fall h in the distance l becomes known. As this fall is often small, it is very important that every precaution be taken to avoid error in the measurements, and that a number of them be taken in order to secure a precise mean. Care should be observed that the stage of water is not varying while these observations are being made, and for this and other purposes a permanent gauge must be established. It is also very important that the points upon the water surface which are selected for comparison should be situated so as to be free from local influences such as eddies, since these often cause marked deviations from the normal surface of the stream. If hook gauges can be used for referring the water levels to the benches probably the most accurate results can be obtained. It has been observed that the surface of a swiftly flowing stream is

not a plane, but a cylinder, which is concave to the bed, its highest elevation being where the velocity is greatest, and hence the two points of reference should be located similarly with respect to the axis of the current. In spite of all precautions, however, the relative error in h will usually be large in the case of slight slopes, unless l be very long, which cannot often occur in streams under conditions of uniformity.

Owing to the uncertainty of determinations of discharge made in the manner just described, the common practice is to gauge the stream by velocity observations, to which subject therefore a large part of this chapter will be devoted. The methods given are equally applicable to conduits and canals, and in Art. 115 will be found a summary which briefly compares the various processes.

Prob. 132. Which has the greater discharge—a stream 2 feet deep and 85 feet wide on a slope of 1 foot per mile, or a stream 3 feet deep and 40 feet wide on a slope of 2 feet per mile?

ARTICLE 107. VELOCITIES IN A CROSS-SECTION.

The mean velocity v is the average of all the velocities of all the small sections or filaments in a cross-section (Art. 93). Some of these individual velocities are much smaller, and others materially larger, than the mean velocity. Along the bottom of the stream, where the frictional resistances are the greatest, the velocities are the least; along the centre of the stream they are the greatest. A brief statement of the general laws of variation of these velocities is now to be made.

In Fig. 73 there is shown at A a cross-section of a stream with contour curves of equal velocity; here the greatest velocity is seen to be near the deepest part of the section a short

distance below the surface. At B is shown a plan of the stream with arrows roughly representing the intensities of the surface velocities at different points; the greatest of these is seen to be near the deepest part or axis of the channel while the others diminish toward the banks, the law of variation being a curve resembling a parabola. At C is shown by arrows the variation of velocities in a vertical line, the smallest being

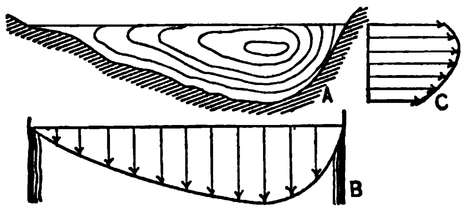


FIG. 73.

at the bottom, and the largest a short distance below the surface; concerning this curve there has been much contention, but it is commonly thought to be a parabola whose axis is horizontal. These are the general laws of the variation of velocity throughout the cross-section; the particular relations are of a complex character, and vary so greatly in channels of different kinds that it is difficult to formulate them, although many attempts to do so have been made. Some of these formulas which connect the mean velocity with particular velocities, such as the maximum surface velocity, mid depth velocity in the axis of the stream, etc., will be given in the following articles in connection with the subject of gauging rivers.

In a straight channel whose bed is of a uniform nature the deepest part is near the middle of its width, while the two sides are approximately symmetrical. In a river bend, however, the deepest part is near the outer bank, while on the inner side the water is shallow: the cause of this is undoubtedly due to the centrifugal force of the current, which, resisting the

change in direction, creates currents which scour away the outer bank or prevents deposits from there occurring. It is well known to all, that rivers of the least slope have the most bends; perhaps this is due to the greater relative influence of such cross currents (Art. 131).

The theory of the flow of water in channels, like that of flow in pipes, is based upon the supposition of a mean velocity which is the average of all the parallel individual velocities in the cross-section. But in fact there are numerous sinuous motions of particles from the bottom to the surface which also consume a portion of the lost head. The influence of these sinuosities is as yet but little understood; when in the future this becomes known a better theory may be possible.

Prob. 133. Find the approximate discharge of a stream whose width is 200 feet, depth 3 feet, slope 0.6 feet per mile, when the bottom is very stony and in bad condition.

ARTICLE 108. THE TRANSPORTING CAPACITY OF CURRENTS.

The fact that the water of streams transports large quantities of earthy matter, either in suspension or by rolling it along the bed of the channel, is well known, and has already been mentioned in Article 103. It is now to be shown that 241
the diameters of bodies which can be moved by the pressure of a current vary as the square of its velocity, and their weights vary as the sixth power of the velocity.

When water causes sand or pebbles to roll along the bed of a channel it must exert a force approximately proportional to the square of the velocity and to the area exposed (Art. 32), or if d be the diameter of the body and α a constant,

$$F = \alpha d^2 v^2.$$

But if motion just occurs, this force is also proportional to the weight of the body, because the frictional resistances of one

body upon another varies as the normal pressure or weight. And as the weight varies as the cube of the diameter,

$$d^3 = \alpha d^2 v^2, \qquad \text{or} \qquad d = \alpha v^2.$$

Now since d varies as v^2, the weight of the body, which is proportional to d^3, must vary as v^6; which proves the proposition as enunciated.

Since the weight of sand and stones when immersed in water is only about one-half their weight in air, the frictional resistances to their motion are slight, and this helps to explain the circumstance that they are so easily transported by currents of moderate velocity. It is found by observation that a pebble about one inch in diameter is rolled along the bed of a channel when the velocity is about 3½ feet per second; hence, according to the above theoretical deduction, a velocity 5 times as great, or 17½ feet per second, will carry along stones of 25 inches diameter. This law of the transporting capacity of flowing water is only an approximate one, for the recorded experiments seem to indicate that the diameters of moving pebbles on the bed of a channel do not vary quite as rapidly as the square of the velocity. The law, moreover, is applicable only to bodies of similar shape, and cannot be used for comparing round pebbles with flat spalls.

The following table gives the velocities on the bed or bottom of the channel which are required to move the materials stated. The corresponding approximate mean velocities in the cross-section given in the last column are derived from the empirical formula deduced by DARCY,

$$v = v' + 11\sqrt{rs},$$

in which v' is the bottom and v the mean velocity. The bottom or transporting velocities were deduced by DUBUAT from experiments in small troughs, and hence are probably slightly

less than the velocities which would move the same materials in channels of natural earth.

	Bottom velocity.	Mean velocity.
Clay, fit for pottery, ...	0.3	0.4
Sand, size of anise-seed, ...	0.4	0.5
Gravel, size of peas, ...	0.6	0.8
Gravel, size of beans, ...	1.2	1.6
Shingle, about 1 inch in diameter, . . .	2.5	3.5
Angular stones, about 1½ inches diameter,	3.5	4.5

The general conclusion to be derived from these figures is that ordinary small, loose earthy materials will be transported or rolled along the bed of a channel by velocities of 2 or 3 feet per second. It is not necessarily to be inferred that this movement of the materials is of an injurious nature in streams with a fixed regimen, but in artificial canals the subject is one that demands close attention. The velocity of the moving objects after starting has been found to be usually less than half that of the current.*

Prob. 134. A stone weighing 0.5 pounds is moved by a current of 3 feet per second; what weight will be moved by a current of 9 feet per second?

ARTICLE 109. THE CURRENT METER.

The most convenient way of precisely measuring the discharge of a canal, conduit, or small stream is by means of a weir which is specially built for that purpose. The flow of a very large conduit or of a large stream cannot, however, be successfully gauged in this manner, both on account of the expense of the dam and weir, and because the weir coefficients are not well known for depths of water greater than about 1.5

* See paper by HERSCHEL on the erosive and abrading power of water, in Journal of the Franklin Institute, May, 1878.

feet. Large quantities of water, therefore, are usually measured by observing the velocity of its flow, and the current meter furnishes a method of doing this which is extensively used, and which gives accurate results.

The current meter is like a windmill, having three or more vanes mounted on a spindle, and so arranged that the face of the mill or wheel always stands normal to the current, the pressure of which causes it to revolve. The number of revolutions of the wheel is approximately proportional to the velocity of the current. In the best forms of instruments the number of revolutions made in a given time is determined by an apparatus on shore or in a boat from which wires lead to the meter under water; at every revolution an electric connection is made and broken which affects a dial on the recording apparatus. The observer has hence only to note the time of beginning and ending of the experiment, and to read the number of revolutions which have occurred during the interval. For a canal or small stream the meter is best operated from a bridge; in large streams a boat must be used.

To derive the velocity from the number of recorded revolutions per second, the meter must be first rated by pushing it at a known velocity in still water. For this purpose a base line several hundred feet long is laid out on shore, and ranges established so that a boat may be rowed over the same distance and the time of its passage be determined. The current meter is placed in the bow of the boat, and a start made sufficiently far from the base so that a uniform velocity can be acquired before reaching it; the distance is then traversed with this uniform velocity and the times observed, as also the actual records of the meter. It is usually found that the number of revolutions are not exactly proportional to the actual velocities of the boat, and hence it is necessary to run the boat

at different velocities per second and ascertain the corresponding number of revolutions of the wheel for each. A table may then be prepared which gives the velocity corresponding to the revolutions per second, from which in subsequent field work the reductions can readily be made. The relation between the velocity V and the number of revolutions per second n can also be expressed by an equation of the form

$$V = \alpha + \beta n + \gamma n^2,$$

and the experiments furnish the data from which the coefficients α, β, and γ can be determined by the help of the Method of Least Squares. For ordinary ranges of velocity γ is usually a small quantity, and it is often taken as zero.

A current meter cannot be used for determining the velocity in a small trough, for the introduction of it into the cross-section would contract the area and cause a change in the velocity in front of the wheel. In large conduits, canals, and rivers it is, however, one of the most convenient and accurate instruments. By holding it at a fixed position below the surface the velocity at that point is found; by causing it to descend at a uniform rate from surface to bottom the mean velocity in that vertical is obtained; and by passing it at a uniform rate over all parts of the cross-section of a channel the mean velocity v is directly determined. It is usually mounted at the end of a long pole, which is graduated so that the depth of the meter below the water surface can be directly read.*

Prob. 135. By rating a certain water meter, the equation $V = 0.159 + 1.905n$ was deduced for velocities varying from 1 to 7 feet per second. Compute the velocity of the current when the wheel revolves 101 times in 41 seconds.

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* See paper by STEARNS in Transactions American Society of Civil Engineers, 1883, vol. xii. p. 301, for detailed account of the use of the current meter in the Sudbury conduit.

ARTICLE 110. FLOATS.

The method for measuring the discharge of streams which has been most extensively used is by observing the velocity of flow by the help of floats. Of these there are three kinds, surface floats, double floats, and rod floats. Surface floats should be sufficiently submerged so as to thoroughly partake of the motion of the upper filaments, and should be made of such a form as not to be readily affected by the wind. The time of their passage over a given distance is determined by two observers at the ends of a base on shore by stop-watches; or only one watch may be used, the instant of passing each section being signalled to the time-keeper. If l be the length of the base, and t the time of passage in seconds, the velocity of the float per second is

$$v = \frac{l}{t}.$$

The numerical work of division can here, as in other cases, be best performed by taking the reciprocal of t from a table, and multiplying it by l, which for convenience may be an even number, such as 100 or 200 feet.

A sub-surface float consists of a small surface float connected by a fine cord or wire with the large real float which is weighted so as to remain submerged, and keep the cord reasonably taut. The surface float should be made of such a form as to offer but slight resistance to the motion, while the lower float is large, it being the object of the combination to determine the velocity of the lower one alone. This arrangement has been extensively used, but it is probable that in all cases the velocity of the large float is somewhat affected by that of the upper one, as well as by the friction of the cord. In general the use of these floats is not to be encouraged, if any other method of measurement can be devised.

$\left[1.012-0.116\sqrt{\frac{d-I}{d}}\right]$ COMPILED BY ALEX [illegible]

$\frac{d-I}{d}$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0		1.000	.995	.992	.988	.985	.983	.981	.979	.977
.1	.975	.973	.971	.970	.968	.967	.965	.964	.962	.961
.2	.960	.958	.957	.956	.955	.954	.953	.951	.950	.949
.3	.948	.947	.946	.945	.944	.943	.942	.941	.940	.939
.4	.938	.937	.936	.936	.935	.934	.933	.932	.931	.930
0.5	.930	.929	.928	.928	.927	.926	.925	.924	.924	.923
.6	.922	.921	.921	.920	.919	.919	.918	.917	.916	.916
.7	.915	.914	.914	.913	.912	.912	.911	.910	.910	.909
.8	.908	.908	.907	.906	.906	.905	.904	.904	.903	.903
.9	.902	.901	.901	.900	.899	.899	.898	.898	.897	.897

FRANCIS FORMULA FOR REDUCING ROD FLOAT VELOCITIES TO MEAN FLOAT VEL.

$V_m = V_r\left[1.012-0.116\sqrt{\frac{d-I}{d}}\right]$; THE TABLE GIVES VALUE OF $\left[1.012-0.116\sqrt{\frac{d-I}{d}}\right]$ FOR CORRESPONDING VALUES OF $\frac{d-I}{d}$, V_m = MEAN VELOCITY; V_r = OBSERVED VELOCITY OF ROD. d = TOTAL DEPTH ; I = IMMERSION OF ROD FLOAT

EXAMPLE: $\frac{d-I}{d} = \frac{12-10}{12} = 0.17$, THEN $\left[1.012-0.116\sqrt{\frac{d-I}{d}}\right] = .964$ AND $V_m = 0.964\,V_r$.

The rod float is a hollow cylinder of tin, which can be weighted by dropping in pebbles or shot so as to stand vertically at any depth. When used for velocity determinations they are weighted so as to reach nearly to the bottom of the channel, and the time of passage over a known distance determined as above explained. It is often stated that the velocity of a rod float is the mean velocity of all the filaments in the vertical plane in which it moves. Theoretically this is not the case; and experiments by FRANCIS have proved that the velocity of the rod is usually from 1 to 5 per cent less than that of the mean velocity in the vertical. FRANCIS has also deduced the following empirical formula for finding the mean velocity V_m from the observed velocity V_r of the rod,

$$V_m = V_r\left(1.012 - 0.116\sqrt{\frac{d'}{d}}\right),$$

in which d is the total depth of the stream, and d' the depth of water below the bottom of the rod.* This expression is probably not a valid one, unless d' is less than about one-quarter of d; usually it will be best to have d' as small as the character of the bed of the channel will allow.

The log used by seamen for ascertaining the speed of vessels may be often conveniently used as a surface float when rough determinations only are desired, it being thrown from a boat or bridge. The cord of course must be previously stretched when wet, so that its length may not be altered by the immersion; if graduated by tags or knots in divisions of six feet, the log may be allowed to float for one minute, and then the number of divisions run out in this time will be ten times the velocity in feet per second.

The determination of particular velocities in streams by means of floats appears to be simple, but in practice many

* Lowell Hydraulic Experiments, 4th Edition, p. 195.

uncertainties are found to arise, owing to wind, eddies, local currents, etc., so that a number of observations are generally required to obtain a precise mean result. For conduits, canals, and for many rivers the use of a current meter will be found to be more satisfactory and less expensive if many observations are required.

Prob. 136. A rod float runs a distance of 100 feet in 42 seconds, the depth of the stream being 6 feet, while the foot of the rod is 6 inches above the bottom. Compute the mean velocity in the vertical.

ARTICLE 111. OTHER CURRENT INDICATORS.

PITOT'S tube is an instrument for measuring the velocity of a current by the velocity-head which it will produce. In its simplest form it consists of a bent glass tube as shown in Fig. 74, in which the mouth of the submerged part is placed so as to directly face the current. The water then rises in the vertical part to a distance h above the surface of the flowing stream, and the velocity is approximately equal to $\sqrt{2gh}$. The only advantage of this instrument is that no time observation is necessary; the disadvantages are many, the chief being that the distance h is always very small, so that errors are liable to be made in determining its value. As actually constructed, PITOT'S apparatus generally consists of two tubes placed side by side with their submerged mouths at right angles, so that when one is opposed to the current, as seen in Fig. 74, the other stands normal to it, and the water surface in the latter tube hence is at the same level as that of the stream. Both tubes are provided with cocks which may be closed while the instrument is immersed, and it can be then lifted from the water and the head h be read at leisure.

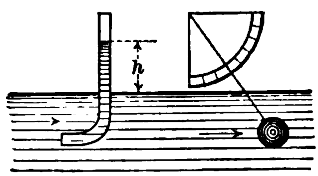

FIG. 74.

It is found that the actual velocity is always less than $\sqrt{2gh}$, and that a coefficient must be deduced for each instrument by moving it in still water at known velocities. PITOT'S tube has been but little used, and is generally regarded as an imperfect instrument for velocity determinations.

The hydrometric pendulum, shown also in Fig. 74, consists of a ball suspended from a string, which by the pressure of the current is kept at a certain inclination from the vertical, the angle of inclination being read on a graduated arc. The relation between this angle and the velocity of the current must be determined experimentally before the instrument can be used in actual observations. This apparatus was employed by some of the early experimenters, but has now gone out of use.

The hydrometric balance is similar in principle to the pendulum, the string being replaced by a rigid rod which is connected with a lever at its upper end, upon which weights are hung so as to keep the rod in a vertical position. The weights measure the intensity of the pressure of the current, and hence its velocity, the relation between them being first experimentally established for each instrument. The hydrometric balance is a mere curiosity, and has never been practically used for velocity determinations. A torsion balance, in which the pressure of the current on a submerged plate causes a spring to be tightened, has also been devised. All the instruments mentioned in this article are adapted only to the measurement of velocities in small troughs or channels, and even for these have been but little used.

Prob. 137. If the head h in a PITOT tube is 0.01 feet, what is the approximate velocity of the current? If an error of 25 per cent be made in reading h, how does this affect the deduced value of the velocity?

ARTICLE 112. GAUGING THE FLOW.

The most common method of gauging the flow of a stream which is too large to be measured by a weir will now be explained. It involves field operations which, although simple in statement, generally require considerable care and expense. In all cases the first step should be to establish a water gauge whose zero is located with reference to a permanent bench mark, so that the stage of water at any time may be determined. Such a gauge is usually graduated to tenths of feet, intermediate values being estimated to hundredths.

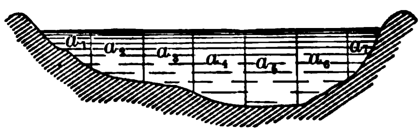

FIG. 75.

One or more sections at right angles to the direction of the current are to be established, and soundings taken at intervals across the stream upon them, the water gauge being read while this is done. The distances between the places of sounding are measured either upon a cord stretched across the stream, or by other methods known to surveyors. The data are thus obtained for obtaining the areas a_1, a_2, a_3, etc., shown upon Fig. 75, and the sum of these is the total area a. Levels should be run out upon the bank beyond the water's edge, so that in case of a rise of the stream the additional areas can be deduced. If a current meter is used, but one section is needed; if floats are used, at least two are required, and these must be located at a place where the channel is of as uniform size as possible.

The mean velocities v_1, v_2, v_3, etc., in each of the sections are next to be determined for each of the sub-areas. If a current meter is used, this may be done by starting at one side of a subdivision, and lowering it at a uniform rate until the bottom is nearly reached, then moving it a foot or two horizontally and raising it to the surface, and continuing until the area has been covered. The velocity then deduced from the whole number

of revolutions is the mean velocity for the subdivision. Or the meter may be simply raised and lowered in a vertical at the middle of the sub-area, and the result will be a close approximation to the mean velocity. If rod floats are used they are started above the upper section, and the times of passing to the lower one noted, as explained in Art. 110, the velocity deduced from a float at the middle of a sub-area being taken as the mean for that area. It will be found that the rod floats are more or less affected by wind, whose direction and intensity should hence always be noted.

The discharge of the stream is the sum of the discharges through the several sub-areas, or

$$q = a_1v_1 + a_2v_2 + a_3v_3 + \text{etc.};$$

and if this be divided by the total area a, the mean velocity for the entire section is determined.

The following notes give the details of a gauging of the Lehigh River, near Bethlehem, Pa., made Oct. 15, 1885, in the above manner by the use of rod floats.* The two sections were 100 feet apart, divided into 10 equal divisions, each 30 feet in width, except the one at the north bank, which was 32 feet. In the second column are given the soundings in feet, in the

Subdivisions.	Depths.	Areas.	Times.	Velocities.	Discharges.
	0.0				
1		55.5	380	0.263	14.6
	3.0				
2		148.5	220	0.454	67.4
	6.0				
3		201.7	185	0.540	108.9
	7.1				
4		217.5	120	0.833	181.2
	7.0				
5		210.0	145	0.690	144.9
	7.0				
6		186.0	150	0.667	124.1
	5.3				
7		150.8	165	0.606	91.4
	4.3				
8		114.0	200	0.500	57.0
	3.0				
9		84.0	320	0.313	26.3
	2.2				
10		42.0	430	0.233	9.8
	0.0				
		$a = 1410.0$			$q = 825.6$

* Journal of Engineering Society of Lehigh University, 1885, vol. i. p. 75.

third the areas in square feet, in the fourth the times of passage of the floats in seconds, in the fifth the velocities in feet per second, which are directly deduced from the times without applying the correction indicated in Art. 110, and in the last are the products a_1v_1, a_2v_2, which are the discharges for the subdivisions a_1, a_2, etc. The total discharge is found to be 826 cubic feet per second, and the mean velocity is

$$v = \frac{826}{1410} = 0.59 \text{ feet per second.}$$

A second gauging of the stream, made a week later, when the water level was 0.59 feet higher, gave for the discharge 1336 cubic feet per second, for the total area 1630 square feet, and for the mean velocity 0.82 feet per second. These results for discharge and velocity should probably be increased about 3 per cent, in order to allow for the difference between the velocities as observed by the rod floats, and the true mean velocities in the middle of the sub-areas.

As to the accuracy of the above method, it may be said that with ordinary work, using rod floats, the discrepancies in results obtained under different conditions ought not to exceed 10 per cent; and with careful work, using current meters, they may often be of a much higher degree of precision. In any event the results derived from such gaugings of rivers are more reliable than can be obtained by any other method.

Prob. 138. Compute the mean depth and the hydraulic radius for the above section of the Lehigh River.

ARTICLE 113. GAUGING BY SURFACE VELOCITIES.

If by any means the mean velocity v of a stream can be found, the discharge is known from the relation $q = av$, the area a being measured as explained in the last article. An approximate value of v may be ascertained by one or more float

measurements by means of the known relations between it and the surface velocities.

The ratio of the mean velocity v to the maximum surface velocity V has been found to usually lie between 0.7 and 0.85, and about 0.8 appears to be a rough mean value. Accordingly,

$$v = 0.8V;$$

from which, if V be accurately determined, v can be computed with an uncertainty usually less than 20 per cent.

Many attempts have been made to deduce a more reliable relation between v and V. The following rule derived from the investigations of BAZIN makes the relation dependent on the coefficient c, whose value for the particular stream is to be obtained from the evidence presented in the last chapter:

$$v = \frac{V}{1 + \frac{25}{c}}.$$

It is probable, however, that the relation depends more on the hydraulic radius and the shape of the section than upon the degree of roughness of the channel, which c mainly represents.

The ratio of the mean velocity v_1 in any vertical to its surface velocity V_1 is less variable, lying between 0.85 and 0.92, so that

$$v_1 = 0.9V_1$$

may be used with but an uncertainty of a few per cent. If several velocities V_1, V_2, etc., be determined by surface floats, the mean velocities v_1, v_2, etc., for the several sub-areas a_1, a_2, etc., are known, and the discharge is $q = a_1v_1 + a_2v_2 +$ etc., as before explained. This method will usually prove unsatisfactory as compared with the use of rod floats.

Since the maximum surface velocity is greater than the mean velocity v, and since the velocities at the shores are

usually very small, it follows that there are in the surface two points at which the velocity is equal to v. If by any means the location of either of these could be discovered, a single velocity observation would give directly the value of v. The position of these points is subject to so much variation in channels of different forms, that no satisfactory method of locating them has yet been devised.

The influence of wind upon the surface velocities is so great, that these methods of determining v will prove useless, except in calm weather. A wind blowing up stream decreases the surface velocities, and one blowing down stream increases them, without materially affecting the mean velocity and discharge.

Prob. 139. A stream 60 feet wide is divided into three sections, having the areas 32, 65, and 38 square feet, and the surface velocities near the middle of these are found to be 1.3, 2.6, and 1.4 per second. What is the approximate mean velocity of the stream?

ARTICLE 114. GAUGING BY SUB-SURFACE VELOCITIES.

By means of a sub-surface float, or by a current meter, the velocity V' at mid-depth in any vertical may be measured. The mean velocity v_1 in that vertical is very closely

$$v_1 = 0.98 V'.$$

In this manner the mean velocities in several verticals across the stream may be determined by a single observation at each point, and these may be used, as in Art. 112, in connection with the corresponding areas to compute the discharge.

It was shown by the observations of HUMPHREYS and ABBOT on the Mississippi that the velocity V' is practically unaffected by wind, the vertical velocity curves for different intensities of wind intersecting each other at mid-depth. The mid-depth velocity is therefore a reliable quantity to determine and use,

particularly as the corresponding mean velocity v_1 for the vertical rarely varies more than 1 or 2 per cent from the value $0.98V'$.

The following relations between velocities in the cross-section were also deduced by HUMPHREYS and ABBOT.* The curve of velocities in any vertical was found to be a parabola whose mean equation is

$$V = 3.26 - 0.7922\left(\frac{y}{d}\right)^2,$$

in which V is the velocity at any distance y above or below the horizontal axis of the parabola, and d is the depth of the water at the point considered; the axis being at the distance $0.297d$ below the surface. The depth of the axis was found to vary greatly with the wind, an up-stream wind of force 4 depressing it to mid-depth, and a down-stream wind of force 5.3 elevating it to the surface. The velocity V at any depth d' was shown to be related to the maximum velocity V_m in that vertical by the equation

$$V = V_m - (bv)^{\frac{1}{2}}\left(\frac{d'-y}{d}\right),$$

in which v is the mean velocity for the entire cross-section, and

$$b = \frac{1.69}{\sqrt{d+1.5}}.$$

These relations and many others which were deduced are very interesting, but are of little value in the actual gauging of streams.

Prob. 140. Show that the vertical velocity formula of HUMPHREYS and ABBOT can be put in the form

$$V = 3.19 + 0.471\frac{x}{d} - 0.792\left(\frac{x}{d}\right)^2,$$

in which x is the depth below the surface.

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* Physics and Hydraulics of the Mississippi River, 2d Edition, 1876.

ARTICLE 115. COMPARISON OF METHODS.

This chapter, together with those preceding, furnishes many methods by which the quantity of water flowing through an orifice, pipe or channel, may be determined. A few remarks may now properly be made by way of summary.

The method of direct measurement in a tank is always the most accurate, but except for small quantities is expensive, and for large quantities is impracticable. Next in reliability and convenience come the methods of gauging by orifices and weirs. An orifice one foot square under a head of 25 feet will discharge about 40 cubic feet per second, which is as large a quantity as can be usually profitably passed through a single opening. A weir 20 feet long with a depth of 2.0 feet will discharge about 200 cubic feet per second, which may be taken as the maximum quantity that can be conveniently thus gauged. The number of weirs may be indeed multiplied for larger discharges, but this is usually forbidden by the expense of construction. Hence for larger quantities of water indirect methods of measurement must be adqpted.

The formulas deduced for the flow in pipes and channels in Chaps. VII and VIII enable an approximate estimation of their discharge to be determined when the coefficients and data which they contain can be closely determined. The remarks in Art. 106 indicate the difficulty of ascertaining these data for streams, and show that the value of the formulas lies in their use in cases of investigation and design rather than for precise gaugings. For small pipes an accurately rated water meter is a cheap and convenient method of measuring the discharge, while for large pipes it will often be found difficult to devise an accurate and economical plan for precise determinations, unless the conditions are such that the discharge may be made to pass over a weir or be retained in a large reservoir

whose capacity is known for every tenth of a foot in depth. For large aqueducts and for canals and streams the only available methods are those explained in this chapter.

Surface floats are not to be recommended except for rude determinations, because they are affected by wind, and because the deduction of mean velocities from them is always subject to much uncertainty. Nevertheless many cases arise in practice where the results found by the use of surface floats are sufficiently precise to give valuable information concerning the flow of streams.

The double float for sub-surface velocities is used in deep and rapid rivers, where a current meter cannot be well operated on account of the difficulty of anchoring a boat. In addition to its disadvantages already mentioned may be noted that of expense, which becomes large when many observations are to be taken.

The method of determining the mean velocities in vertical planes by rod floats is very convenient in canals and channels which are not too deep or too shallow. The precision of a velocity determination by a rod float is always much greater than that of one taken by the double float, so that the former is to be preferred when circumstances will allow.

Current-meter observations are those which now take the highest rank for precision and rapidity of execution. The first cost of the outfit is greater than that required for rod floats, but if much work is to be done it will prove the cheapest. The main objection is to the errors which may be introduced from the lack of proper rating: this is required to be done at regular intervals, as it is found that the relation between the velocity and the recorded number of revolutions sometimes changes during use.

In the execution of hydraulic operations which involve the measurement of water a method is to be selected which will give the highest degree of precision with a given expenditure, or which will secure a given degree of precision at a minimum expense. Any one can build a road, or a water supply-system; but the art of engineering teaches how to build it well, and at the least cost of construction and maintenance. So the science of hydraulics teaches the laws of flow and records the results of experiments, so that when the discharge of a conduit is to be measured or a stream is to be gauged the engineer may select that method which will furnish the required information in the most satisfactory manner and at the least expense.

Prob. 141. Devise a method for measuring the velocity of a current different from any described in the preceding pages.

ARTICLE 116. VARIATIONS IN VELOCITY AND DISCHARGE.

When the stage of water rises and falls a corresponding increase or decrease occurs in the velocity and discharge. The relation of these variations to the change in depth may be approximately ascertained in the following manner, the slope of the water surface being regarded as remaining uniform: Let the stream be wide, so that its hydraulic radius is nearly equal to the mean depth d; then

$$v = c\sqrt{ds} = cs^{\frac{1}{2}}d^{\frac{1}{2}}.$$

Differentiating this with respect to v and d gives

$$\delta v = \tfrac{1}{2}cs^{\frac{1}{2}}d^{-\frac{1}{2}}\delta d = \tfrac{1}{2}v\frac{\delta d}{d},$$

or

$$\frac{\delta v}{v} = \frac{1}{2}\frac{\delta d}{d}.$$

Here the first member is the relative change in velocity when the depth varies from d to $d \pm \delta d$, and the equation hence

shows that the relative change in velocity is one-half the relative change in depth. For example, a stream 3 feet deep, and with a mean velocity of 2 feet per second, rises so that the depth is 3.3 feet; then

$$\delta v = 2 \times \frac{1}{2} \times \frac{0.3}{3} = 0.1,$$

and the velocity becomes $2 + 0.1 = 2.1$ feet per second. This conclusion is of course the more accurate the smaller the variation δd.

In the same manner the variation in discharge may be found. Thus: let b be the breadth of the stream, then

$$q = cbd\sqrt{ds} = cbs^{\frac{1}{2}}d^{\frac{3}{2}};$$

$$\delta q = \tfrac{3}{2}cbs^{\frac{1}{2}}d^{\frac{1}{2}}\delta d;$$

$$\frac{\delta q}{q} = \frac{3}{2}\frac{\delta d}{d}.$$

Hence the relative change in discharge is $1\frac{1}{2}$ times that of the relative change in depth. This rule, like the preceding, supposes that δd is very small, and will not apply to large variations.

The above conclusions may be expressed as follows: If the mean depth changes 1 per cent, the velocity changes 0.5 per cent, and the discharge changes 1.5 per cent. They are only true for streams with such cross-sections that the hydraulic radius may be regarded as proportional to the depth, and even for such sections are only exact for small variations in d and v. They also assume that the slope s remains the same after the rise or fall as before; this will be the case if a condition of permanency is established, but, as a rule, while the stage of water is rising the slope is increasing, and while falling it is decreasing.

Prob. 142. A stream of 4 feet mean depth delivers 800 cubic feet per second. What will be the discharge when the depth is decreased to 3.9 feet?

ARTICLE 117. NON-UNIFORM FLOW.

In all the cases thus far considered, the slope of the channel, its cross-section, and the depth of the water have been regarded as constant. If these are variable along different reaches of the channel the case is one of non-uniformity, and the preceding discussions do not apply except to the single reaches. The flow being permanent, the same quantity of water passes each section per second, but its velocity and depth vary as the slope and cross-section change. To discuss this case let there be several lengths, $l_1, l_2, \ldots, l_n$, which have the falls $h_1, h_2, \ldots, h_n$, the water sections being $a_1, a_2, \ldots, a_n$, the wetted perimeter $p_1, p_2, \ldots, p_n$, and the velocities $v_1, v_2, \ldots, v_n$. The total fall $h_1 + h_2 + \ldots + h_n$ is expressed by h. Now the head corresponding to the mean velocity in the first section is $\frac{v_1^2}{2g}$. The theoretic head for the last section is $h + \frac{v_1^2}{2g}$, while the actual velocity-head is $\frac{v_n^2}{2g}$. The difference between these is the head lost in friction, or

$$h + \frac{v_1^2}{2g} - \frac{v_n^2}{2g} = f_1 \frac{p_1 l_1}{a_1} \frac{v_1^2}{2g} + f_2 \frac{p_2 l_2}{a_2} \frac{v_2^2}{2g} + \ldots + f_n \frac{p_n l_n}{a_n} \frac{v_n^2}{2g},$$

in which $f_1, f_2, \ldots, f_n$ are the friction factors for the different sections and surfaces, whose values in terms of the velocity coefficient c are, as seen from Art. 94,

$$f_1 = \frac{2g}{c_1^2}, \quad f_2 = \frac{2g}{c_2^2}, \quad \ldots, \quad f_n = \frac{2g}{c_n^2}.$$

Let q be the discharge per second; then, as the flow is permanent,

$$v_1 = \frac{q}{a_1}, \quad v_2 = \frac{q}{a_2}, \quad \ldots, \quad v_n = \frac{q}{a_n}.$$

Inserting in the equation these values of f and v, it becomes

$$h = \frac{q^2}{2g}\left(\frac{1}{a_n^2} - \frac{1}{a_1^2}\right) + q^2\left(\frac{p_1 l_1}{c_1^2 a_1^3} + \frac{p_2 l_2}{c_2^2 a_2^3} + \ldots + \frac{p_n l_n}{c_n^2 a_n^3}\right)$$

which is a fundamental formula for the discussion of the flow in non-uniform channels. Since the values of c given in this chapter are for English feet, the data of numerical problems can be inserted only when expressed in the same unit.

The above discussion shows that the discharge q is a consequence, not only of the total fall h in the entire length of the channel, but also of the dimensions of the various cross-sections. The assumption has been made that a and p are constant in each of the parts considered; this can be realized by taking the lengths l_1, l_2, etc., sufficiently short. If only one part be considered in which a and p are constant, $a_n = a_1$, all the terms but one in the second member disappear, and the two equations reduce to the simple formulas previously deduced for the velocity and discharge in a uniform channel.

An interesting problem is that where the flow is non-uniform in a channel of constant slope and section, which may be caused by an obstruction in the stream above or below the part considered. Here let a_1 and a_2 be two sections whose distance apart is l, and let v_1 and v_2 be the mean velocities in those sections. Then if a and p be average values of the wetted area and perimeter, the formula becomes

$$h = \frac{q^2}{2g}\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) + q^2\frac{pl}{c^2a^3},$$

from which q can be computed when the other quantities are known. The important problem, however, is to discuss the change in depth between the two sections. For this purpose let A_1A_2 in Fig. 76 be the longitudinal profile of the water surface, let A_1D be horizontal, and A_1C be drawn parallel to the bed B_1B_2. The depths A_1B_1 and A_2B_2 are represented by d_1 and d_2, the latter being taken as the larger. Let i be the constant slope of the bed B_1B_2; then $DC = il$, and since $DA_2 = h$ and $A_2C = d_2 - d_1$,

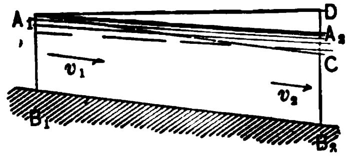

FIG. 76.

$$h = il - (d_2 - d_1).$$

Inserting this value of h in the above equation and solving for l, there results

$$l = \frac{(d_2 - d_1) - \frac{q^2}{2g}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)}{i - \frac{q^2p}{c^2a^3}}, \quad . \quad . \quad . \quad . \quad (76)$$

from which the length l corresponding to a change in depth $d_2 - d_1$ can be approximately computed. This formula is the more accurate the shorter the length l, since then the mean quantities p and a can be obtained with greater precision, and c is subject to less variation.

The inverse problem, to find the change in depth when l is given, cannot be directly solved by this formula, because the areas are functions of the depths. If the change is not great, however, a solution may be effected for the case of a channel whose breadth b is constant by regarding p and a as equal to p_1 and a_1; and also by putting

$$\frac{1}{a_1^2} - \frac{1}{a_2^2} = \frac{a_2^2 - a_1^2}{a_1^2a_2^2} = \frac{d_2^2 - d_1^2}{b^2d_1^2d_2^2} = \frac{(d_2+d_1)(d_2-d_1)}{b^2d_1^4} = \frac{2(d_2-d_1)}{b^2d_1^3}.$$

The formula then becomes

$$\frac{d_2 - d}{l} = \frac{i - \dfrac{q^2 p_1}{c^2 a_1^3}}{1 - \dfrac{q^2}{g b^2 d_1^3}}, \quad . \quad . \quad . \quad . \quad . \quad (76)'$$

from which d_2 can be approximately computed when all the other quantities are given.

Fig. 76 is drawn for the case of depth increasing down stream, but the reasoning is general, and the formulas apply equally well when the depth decreases. In the latter case the point A_2 is below C, and $d_2 - d_1$ will be found to be negative. As an example, let it be required to determine the decrease in depth in a rectangular conduit 5 feet wide and 333 feet long, which is laid with its bottom level, the depth of water at the entrance being maintained at 2 feet, and the quantity supplied being 20 cubic feet per second. Here $l = 333$, $b = 5$, $d_1 = 2$, $p_1 = 5 + 4 = 9$, $q = 20$, and $i = 0$. Taking $c = 89$, and substituting all values in the formula, there is found $d_2 - d_1 = -0.16$ feet, whence $d_2 = 1.84$ feet, which is to be regarded as an approximate probable value. It is likely that values of $d_2 - d_1$ computed in this manner are liable to an uncertainty of 10 or 20 per cent, the longer the distance l the greater being the error of the formula. In strictness also c varies with depth, but errors from this cause are small when compared to those arising in selecting its value.

Prob. 143. Compute the value of d_2 for the above example when the bed of the conduit has the uniform slope $i = 0.01$.

Ans. $d_2 = 5.38$ feet.

ARTICLE 118. THE SURFACE CURVE.

In the case of uniform flow the slope of the water surface is parallel to that of the bed of the channel, and the longitudinal

profile of the water surface is a straight line. In non-uniform flow, however, the slope of the water surface continually varies, and the longitudinal profile is a curve whose nature is now to be investigated. As in the last article, the slope i of the bed of the channel will be taken as constant, and its cross-section will be regarded as rectangular. Moreover, it will be assumed that the stream is wide compared to its depth, so that the wetted perimeter may be taken as equal to the width and the hydraulic radius equal to the mean depth (Art. 93). These assumptions are closely fulfilled in many canals and rivers.

The last formula of the preceding article is rigidly exact if the sections a_1 and a_2 are consecutive, so that l becomes δl and $d_2 - d_1$ becomes δd. Making these changes, and placing $\frac{p}{a}$ equal to $\frac{1}{d}$, in accordance with the above assumptions, the formula becomes

$$\frac{\delta d}{\delta l} = \frac{i - \frac{q^2}{c^2 b^2 d^3}}{1 - \frac{q^2}{g b^2 d^3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which d is the depth of the water at the place considered. This is the general differential equation of the surface curve.

To discuss this curve let D be the depth of the water if the flow were uniform. The slope s of the water surface would then be equal to the slope i of the bed of the channel, and from the general equation for mean velocity,

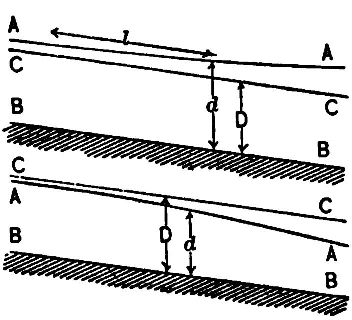

FIG. 77.

$$q = av = cbD\sqrt{ri} = cbD\sqrt{Di}.$$

Inserting this value of q the equation reduces to

$$\frac{\delta d}{\delta l} = i \frac{1 - \frac{D^3}{d^3}}{1 - \frac{c^2 i}{g} \frac{D^3}{d^3}}, \quad \ldots\ldots \quad (77)'$$

in which d and l are the only variables, the former being the ordinate and the latter the abscissa, measured parallel to the bed BB, of any point of the surface curve.

First, suppose that D is less than d, as in the upper diagram of Fig. 77, where AA is the surface curve under the non-uniform flow, and CC is the line which the surface would take in case of uniform flow. The numerator of (77)′ is then positive, and the denominator is also positive, since i is very small. Hence δd is positive, and it increases with d in the direction of the flow; going up stream it decreases with d_1, and the surface curve becomes tangent to CC when $d = D$. This form of curve is that usually produced above a dam, and is called the curve of backwater.

Second, let d be less than D, as in the second diagram of Fig. 74. The numerator is then negative, and the denominator positive; δd is accordingly negative, and AA is concave to the bed BB, whereas in the former case it was convex. This form of surface curve may occur when a sudden fall exists in the stream below the point considered; it is of slight practical importance compared to the previous case.

A very curious phenomenon is that of the so-called "jump" which sometimes occurs in shallow channels, as shown in Fig. 78. This happens when the denominator in (77)′ is zero, the numerator being positive; then $\frac{\delta d}{\delta l}$ becomes infinite, and the

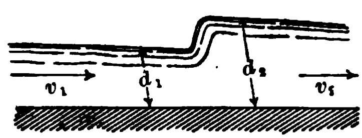

FIG. 78.

water surface stands normal to the bed. Placing the denominator of (77) equal to zero, there is found

$$q^2 = gb^2d^3 \quad \text{or} \quad v^2 = gd.$$

Now by further consideration it will appear that the varying denominator in passing through zero changes its sign. Above the jump where the depth is d_1 the velocity is greater than $\sqrt{gd_1}$, and below it is less than $\sqrt{gd_2}$. The condition for the occurrence of the jump is that an obstruction should exist in the stream below, that the slope i should not be small, and that the velocity should be greater than $\sqrt{gd_1}$. To find the slope i which is necessary,

$$v_1 = c\sqrt{d_1 i} \qquad v_1^2 > gd_1 \qquad \text{whence } i > \frac{g}{c^2}.$$

Hence the jump cannot occur when i is less than $\frac{g}{c^2}$. For an unplaned plank trough c may be taken at about 100; hence the slope for this must be equal to or greater than 0.00322.

To determine the height of the jump, or the value of d_2 in terms of d_1, it is to be observed that the lost head is $\frac{v_1^2}{2g} - \frac{v_2^2}{2g}$, and that this is lost in two ways, first by the impact due to the enlargement of section (Art. 68), and second by the rising of the whole quantity of water through the height $\frac{1}{2}(d_2 - d_1)$, the loss in friction in the short distance between d_1 and d_2 being neglected. Hence

$$\frac{v_1^2 - v_2^2}{2g} = \frac{(v_1 - v_2)^2}{2g} + \frac{d_2 - d_1}{2}.$$

Inserting in this the value of v_2, found from the relation $v_2 d_2 = v_1 d_1$, dividing by $d_2 - d_1$, and solving for d_2, gives

$$d_2 = 2\sqrt{d_1 \frac{v_1^2}{2g}}. \qquad \qquad (78)$$

The following is a comparison between the values of d_2 computed by this formula and the observed values in four experiments made by BIDONE, the depths being in feet:

d_1.	v_1.	Observed d_2.	Computed d_2.
0.149	4.59	0.423	0.439
0.154	4.47	0.421	0.437
0.208	5.59	0.613	0.636
0.246	6.28	0.739	0.777

The agreement is very fair, the computed values being all slightly greater than the observed, which should be the case, because the above reasoning omits the frictional resistances between the points where d_1 and d_2 are measured.

Prob. 144. Discuss formula (78) by placing for q its value $cbd\sqrt{ds}$, where s is the slope of the water surface.

ARTICLE 119. BACKWATER.

When a dam is built across a channel the water surface is raised for a long distance up stream. This is a fruitful source of contention, and accordingly many attempts have been made to discuss it theoretically, in order to be able to compute the probable increase in depth at various distances back from a proposed dam. None of these can be said to have been successful except for the simple case where the slope of the bed of the channel is constant, and its cross-section such that the width may be regarded as uniform and the hydraulic radius be taken as equal to the depth. These conditions are closely fulfilled for many streams, and an approximate solution may be made by the formula (77) of Art. 118. It is desirable, however, to obtain an exact equation of the surface curve, so as to secure a more reliable method.

For this purpose the differential equation (77)′ of the last article may be written in the form

$$\frac{\delta l}{\delta d} = \frac{1}{i}\left(1 + \frac{1 - \frac{c^2 i}{g}}{\frac{d^3}{D^3} - 1}\right),$$

in which l and d are the co-ordinates of any point of the curve. Let $\frac{d}{D}$ be the independent variable x, so that $d = Dx$; then

$$\delta l = \frac{D}{i}\,\delta x + \frac{D}{i}\left(1 - \frac{c^2 i}{g}\right)\frac{\delta x}{x^3 - 1},$$

the general integral of which is

$$l = \frac{Dx}{i} - \frac{D}{i}\left(1 - \frac{c^2 i}{g}\right)\left(\frac{1}{6}\log_e \frac{x^2 + x + 1}{(x - 1)^2} - \frac{1}{\sqrt{3}}\operatorname{arc\,cot}\frac{2x + 1}{\sqrt{3}}\right),$$

which is the equation of the surface curve. To use this let $\phi(x)$ or $\phi\left(\frac{d}{D}\right)$ be put as an abbreviation for the logarithmic and circular function in the second member. Also let d_2 be the depth at the dam, and let l be measured up stream from that point to a section where the depth is d_1. Then taking the integral between these limits the equation becomes

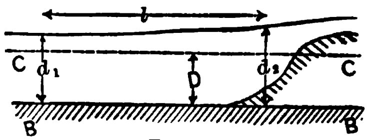

FIG. 79.

$$l = \frac{d_2 - d_1}{i} + D\left(\frac{1}{i} - \frac{c^2}{g}\right)\left[\phi\left(\frac{d_1}{D}\right) - \phi\left(\frac{d_2}{D}\right)\right], \quad . \quad (79)$$

which is the practical formula for use. In like manner, d_2 may represent a given depth at any section, and d_1 any depth farther up the stream.

When $d = D$, or the depth of the backwater becomes equal to that of the previous uniform flow, x is unity, and hence l is

infinity. The slope CC of uniform flow is therefore an asymptote to the backwater curve. Accordingly, no matter how little d_2 may exceed D, the depth d_1 is always greater than D, although it often happens for steep slopes that d_1 becomes practically equal to D at distances above d_2, which are not great. In the investigations of backwater problems there are two cases: d_1 and d_2 may be given and l is to be found, or l is given and one of the depths is to be found. To solve these problems, a table giving values of the backwater function $\phi\left(\frac{d}{D}\right)$ will be found on the next page.* The argument of the table is $\frac{D}{d}$, which being less than unity, is more convenient for tabular purposes than $\frac{d}{D}$, whose values range from 0 to ∞. The following examples will illustrate the method of procedure.

A stream of 5 feet depth is to be dammed so that the water just above the dam will be 10 feet. Its uniform slope is 0.000189, or a little less than one foot per mile, and the surface of its channel is such that the coefficient c is 65. It is required to find the distance back from the dam at which the depth of water is 6 feet. Here $d_2 = 10$, $d_1 = 6$, $D = 5$, $\frac{D}{d_2} = 0.5$ for which the table gives $\phi\left(\frac{d_2}{D}\right) = 0.1318$, $\frac{D}{d_1} = 0.833$ for which the table gives $\phi\left(\frac{d_1}{D}\right) = 0.4792$, and $\frac{1}{i} = 5291$. These values inserted in the formula give

$$l = (10 - 6)5291 + 5\left(5291 - \frac{65^2}{32.16}\right)(0.4792 - 0.1318);$$

$$l = 30\,125 \text{ feet} = 5.70 \text{ miles.}$$

* From BRESSE's La Mécanique Appliquée (Paris, 1873), vol. ii. p. 556.

TABLE XXII. VALUES OF THE BACKWATER FUNCTION.

$\frac{D}{d}$	$\phi\left(\frac{d}{D}\right)$	$\frac{D}{d}$	$\phi\left(\frac{d}{D}\right)$	$\frac{D}{d}$	$\phi\left(\frac{d}{D}\right)$	$\frac{D}{d}$	$\phi\left(\frac{d}{D}\right)$
1.	∞	0.948	0.8685	0.815	0.4454	0.52	0.1435
0.999	2.1834	.946	.8539	.810	.4367	.51	.1376
.998	1.9532	.944	.8418	.805	.4281	.50	.1318
.997	1.8172	.942	.8301	.800	.4198	.49	.1262
.996	1.7213	.940	.8188	.795	.4117	.48	.1207
.995	1.6469	.938	.8079	.790	.4039	.47	.1154
.994	1.5861	.936	.7973	.785	.3962	.46	.1102
.993	1.5348	.934	.7871	.780	.3886	.45	.1052
.992	1.4902	.932	.7772	.775	.3813	44	.1003
.991	1.4510	.930	.7675	.770	.3741	.43	.0995
.990	1.4159	.928	.7581	.765	.3671	.42	.0909
.989	1.3841	.926	.7490	.760	.3603	.41	.0865
.988	1.3551	.924	.7401	.755	.3526	.40	.0821
.987	1.3248	.922	.7315	.750	.3470	.39	.0779
.986	1.3037	.920	.7231	.745	.3406	.38	.0738
.985	1.2807	.918	.7149	.740	.3343	.37	.0699
.984	1.2592	.916	.7069	.735	.3282	.36	.0666
.983	1.2390	.914	.6990	.730	.3221	.35	.0623
.982	1.2199	.912	.6914	.725	.3162	.34	.0587
.981	1.2019	.910	.6839	.720	.3104	.33	.0553
.980	1.1848	.908	.6766	.715	.3047	.32	.0519
.979	1.1686	.906	.6695	.710	.2991	.31	.0486
.978	1.1531	.904	.6625	.705	.2937	.30	.0455
.977	1.1383	.902	.6556	.70	.2883	.29	.0425
.976	1.1241	.900	.6489	.69	.2778	.28	.0395
.975	1.1105	.895	.6327	.68	.2677	.27	.0367
.974	1.0974	.890	.6173	.67	.2580	.26	.0340
.973	1.0848	.885	.6025	.66	.2486	.25	.0314
.972	1.0727	.880	.5884	.65	.2395	.24	.0290
.971	1.0610	.875	.5749	.64	.2306	.23	.0266
.970	1.0497	.870	.5619	.63	.2221	.22	.0243
.968	1.0282	.865	.5494	.62	.2138	.21	.0221
.966	1.0080	.860	.5374	.61	.2058	.20	.0201
.964	0.9890	.855	.5258	.60	.1980	.18	.0162
.962	.9700	.850	.5146	.59	.1905	.16	.0128
.960	.9539	.845	.5037	.58	.1832	.14	.0098
.958	.9376	.840	.4932	.57	.1761	.12	.0072
.956	.9221	.835	.4831	.56	.1692	.10	.0050
.954	.9073	.830	.4733	.55	.1625	.06	.0018
.952	.8931	.825	.4637	.54	.1560	.01	.0001
.950	.8795	.820	.4544	.53	.1497	.00	.0000

In this case the water is raised one foot at a distance of 5.7 miles up stream from the dam, in spite of the fact that the fall in the bed of the channel is nearly 5.7 feet.

The inverse problem, to compute d_1 or d_2, when l and d_2 or d_1 is given, can only be solved by repeated tentative trials by the help of Table XXII. For example, let $l = 30\,125$ feet, the other data as above, and it be required to determine d_2 so that d_1 shall be only 5.2 feet, or 0.2 feet greater than the original depth of 5 feet. Here $\frac{D}{d_1} = \frac{5}{5.2} = 0.962$, and from the table $\phi\left(\frac{d_1}{D}\right) = 0.9700$. Then the formula becomes

$$30\,125 = (d_2 - 5.2)5291 + 5 \times 5160\left[0.9700 - \phi\left(\frac{d_2}{D}\right)\right],$$

which reduces to

$$32\,610 = 5291 d_2 - 25\,800\phi\left(\frac{d_2}{D}\right).$$

Values of d_2 are now to be assumed until one is found which satisfies this equation. Let $d_2 = 8$ feet, then $\frac{D}{d_2} = 0.625$, and from the table $\phi\left(\frac{d_2}{D}\right) = 0.2179$. Substituting these,

$$32\,610 = 42\,328 - 5\,622 = 36\,706,$$

which shows that the assumed value is too large. Again, take $d_2 = 7$ feet, then $\frac{D}{d_2} = 0.714$, and from the table $\phi\left(\frac{d_2}{D}\right) = 0.3036$, whence

$$32\,610 = 37\,037 - 7\,833 = 29\,204,$$

which shows that 7 feet is too small. If $d_2 = 7.4$ feet,

$$\frac{D}{d_2} = 0.675 \quad \text{and} \quad \phi\left(\frac{d_2}{D}\right) = 0.2628,$$

and then

$$32\,610 = 39\,153 - 6\,780 = 32\,373.$$

This indicates that 7.4 is a little too small, and on trying 7.5 it is found to be too large. The value of d_2 hence lies between 7.4 and 7.5 feet, which is as close a solution as will generally be required. The height of the dam may now be computed by Art. 58, taking the rise d' at about 2.45 feet.

In conclusion, it should be said that if the slope, width, or depth changes materially, the above method cannot be employed in which the distance l is counted from the dam as an origin. In such cases the stream should be divided into reaches, for each of which these quantities can be regarded as constant. The formula can then be used for the first reach, and the depth at its upper section determined; calling this depth d_2, the application can then be made to the next reach, and so on in order. Strictly speaking, the coefficient c varies with the depth, and by KUTTER'S formula (Art. 101) its varying values may be ascertained, if it be thought worth the while. Even if this be done, the resulting computations must be regarded as liable to considerable uncertainty. In computing depths for given lengths probably an uncertainty of 10 per cent or more in values of $d_2 - d_1$ should be expected. In regard to the depth D, it may be said that this should be determined by the actual measurement of the area and wetted perimeter of the cross-section during uniform flow, the hydraulic radius computed from these being taken as D.

Prob. 145. A stream whose cross-section is 2400 square feet and wetted perimeter 300 feet has a uniform slope of 2.07 feet per mile, and its condition is such that $c = 70$. It is proposed to build a dam which raises the water 6 feet above its former level, without increasing its width. Compute the amount of rise due to the backwater at distances of 1, 2, and 3 miles up stream from the dam.

CHAPTER X.

MEASUREMENT OF WATER POWER.

ARTICLE 120. THEORETIC AND EFFECTIVE POWER.

The theoretic energy of W pounds of water falling through h feet is Wh foot-pounds, and if this occurs in one second the energy per second is Wh, and the theoretic horse-power is

$$\overline{HP} = \frac{Wh}{550} = 0.001818\,Wh. \quad . \quad . \quad . \quad . \quad (80)$$

If this power could all be utilized it would be able to lift the same weight of water per second through the same vertical height, and an efficiency of unity would be secured. Owing to friction, impact, leakage, and other losses, the efficiency must always be less than unity.

When the energy of a water-fall is to be transformed into useful work the water is made to pass over a wheel or through a hydraulic motor in such a manner that when the fall h has been accomplished the water has little or no velocity. If the water falls freely through the height h it acquires the velocity $\sqrt{2gh}$, and the energy is still potential, and equal to Wh. If, however, this velocity is destroyed, the energy is either transformed into heat or into useful work. In the case of flow through a long pipe nearly all the energy of the head h may be expended in heat in overcoming the frictional resistances; such a method of bringing water to a motor is therefore to be avoided.

To utilize the energy of a water-fall in work the water is to be collected in a reservoir, canal, or head race, from which it is

carried to the motor through a pipe, penstock, or flume, and after doing its work it issues into the tail race or lower level. In designing these constructions care should be taken to avoid losses in energy or head, and for this purpose the principles of the preceding chapters should be applied. The entrance from the head race into the penstock, and from the penstock to the motor, should be smooth and well rounded; sudden changes in cross-section should be avoided, and all velocities should be low except that which is to be utilized in impulse. If these precautions be carefully observed the loss in the head h outside of the motor can be made very small.

The effective power of a water-fall, or that utilized by the motor, may be in the best constructions as high as 90 per cent of the theoretic power. In any case, if e be the efficiency and k the work actually obtained,

$$k = eK = eWh.$$

That hydraulic motor will be the best, other things being equal, which furnishes the highest value of e. In practice values of e are usually between the limits 0.25 and 0.90, the lower values occurring where a cheap and abundant water supply exists, so that sufficient power can be obtained with an inexpensive wheel, for it is a general rule that the cost of a hydraulic motor increases with the efficiency.

There are to be distinguished two efficiencies—the efficiency of the fall and the efficiency of the motor. The same expression

$$e = \frac{k}{Wh}$$

will apply to both, k being the effective work of the motor. The efficiency of the fall is that value of e found by using the actual weight W delivered per second, and the total height h from the water level in the head race to that in the tail race.

The efficiency of the motor is that value of e found by using the actual weight W which passes through the motor, and the effective head h that acts upon it. The second W may be less than the first on account of leakage, and the second h may be less than the first on account of losses of head.

To determine the theoretic power in any case, it is only necessary to measure W and h and insert their values in formula (80). The two following articles will treat of these measurements, and the determination of the effective power of the motor will be discussed afterwards. The efficiency e is, then, the ratio of the effective power to the theoretic power, or the ratio of the effective work to the theoretic work.

Prob. 146. A weir with end contractions and no velocity of approach has a length of 1.33 feet, and the depth on the crest is 0.406 feet. The same water passes through a small turbine under the effective head 10.49 feet. Compute the theoretic horse-power. Ans. 1.28.

ARTICLE 121. MEASUREMENT OF THE WATER.

In order to determine the weight W which is delivered per second there must be known the discharge per second q, and the weight of a cubic unit of water, or

$$W = wq.$$

The quantity w is to be found by weighing very accurately one cubic foot, or any given volume of water, or by noting the temperature and using the table in Art. 3. In common approximate computations w may be taken at 62.5 pounds per cubic foot. In precise tests of motors, however, its actual value should be ascertained as closely as possible.

The measurement of the flow of water through orifices, weirs, tubes, pipes, and channels has been so fully discussed in

the preceding chapters, that it only remains here to mention one or two simple methods applicable to small quantities, and to make a few remarks regarding the subject of leakage. In any particular case that method of determining q is to be selected which will furnish the required degree of precision with the least expense (Art. 115).

For a small discharge the water may be allowed to fall into a tank of known capacity. The tank should be of uniform horizontal cross-section, whose area can be accurately determined, and then the heights alone need be observed in order to find the volume. These in precise work will be read by hook gauges, and in cases of less accuracy by measurements with a graduated rod. At the beginning of the experiment a sufficient quantity of water must be in the tank so that a reading of the gauge can be taken; the water is then allowed to flow in, the time between the beginning and end of the experiment being determined by a stop-watch, duly tested and rated. This time must not be short, in order that the slight errors in reading the watch may not affect the result. The gauge is read at the close of the test after the surface of the water becomes quiet, and the difference of the gauge-readings gives the depth which has flowed in during the observed time. The depth multiplied by the area of the cross-section gives the volume, and this divided by the number of seconds during which the flow occurred furnishes the discharge per second q.

If the discharge be very small, it may be advisable to weigh the water rather than to measure the depths and cross-sections. The total weight divided by the time of flow then gives directly the weight W. This has the advantage of requiring no temperature observation, and is probably the most accurate of all methods, but unfortunately it is not possible to weigh a considerable volume of water except at great expense.

When water is furnished to a motor through a small pipe a water meter may often be advantageously used to determine the discharge. This consists of a box with two chambers, the water entering into one and passing out of the other. In going from the first to the second chamber the water moves a vane, a piston, a disk, or some other device, which communicates motion to a train of clockwork, and thereby causes pointers to move on dials. The external appearance of a water meter is similar to that of a gas meter, and it is read in the same way. No water meter, however, can be regarded as accurate until it has been tested by comparing the discharge as recorded by it with the actual discharge as determined by measurement or weighing in a tank. Such a test furnishes the constants for correcting the result found by its readings, which otherwise is liable to be 5 or 10 per cent in error.

The leakage which occurs in the flume or penstock before the water reaches the wheel should not be included in the value of W, which is used in computing its efficiency. The manner of determining the amount of leakage will vary with the particular circumstances of the case in hand. If it be very small, it may be caught in pails and directly weighed. If large in quantity, the gates which admit water to the wheel may be closed, and the leakage being then led into the tail race it may be there measured by a weir, or by allowing it to collect in a tank. The leakage from a vertical penstock whose cross-section is known may be ascertained by filling it with water, the wheel being still, and then observing the fall of the water level at regular intervals of time. In designing constructions to bring water to a motor, it is best, of course, to arrange them so that all leakage will be avoided, but this cannot often be fully attained, except at great expense.

The most common method of measuring q is by means of a weir placed in the tail race below the wheel. This has the

disadvantage that it sometimes lessens the fall which would be otherwise available, and that often the velocity of approach is high. It has, however, the advantage of cheapness in construction and operation, and for any considerable discharge appears to be almost the only method which is both economical and precise.

Prob. 147. A vertical penstock whose cross-section is 15.98 square feet is filled with water to a depth of 10.50 feet. During the space of two minutes the water level sinks 0.02 feet. What is the leakage in cubic feet per second?

ARTICLE 122. MEASUREMENT OF THE HEAD.

The total available head h between the surface of the water in the reservoir or head race and that in the lower pool or tail race is determined by running a line of levels from one to the other. Permanent bench marks being established, gauges can then be set in the head and tail races, and graduated so that their zero points will be at some datum below the tail-race level. During the test of a wheel each gauge is read by an observer at stated intervals, and the difference of the readings gives the head h. In some cases it is possible to have a floating gauge on the lower level, the graduated rod of which is placed alongside of a glass tube that communicates with the upper level; the head h is then directly read by noting the point of the graduation which coincides with the water surface in the tube. This device requires but one observer, while the former requires two; but it is usually not the cheapest arrangement unless a large number of observations are to be taken.

When water is delivered through a nozzle or pipe to a hydraulic motor the head which is to be determined for ascertaining the efficiency of the motor is not the total fall, since a large part of that may be lost in friction in the pipe, but is merely

the velocity-head $\frac{v^2}{2g}$ of the issuing jet. The value of v is known when the discharge q and the area of the cross-section of the stream have been determined, and

$$h = \frac{v^2}{2g} = \frac{q^2}{2ga^2}.$$

In the same manner when a stream flows in a channel against the vanes of an undershot wheel the effective head is the velocity-head, and the theoretic energy is

$$K = Wh = W\frac{v^2}{2g} = \frac{wq^3}{2ga^2}.$$

If, however, the water in passing through the wheel falls a distance h_0 below the mouth of the nozzle, then the effective head is

$$h = \frac{v^2}{2g} + h_0.$$

In order to fully utilize the fall h_0 it is plain that the wheel should be placed as near the level of the tail race as possible.

When water enters upon a wheel through an orifice which is controlled by a gate, losses of head will result, which can be estimated by the rules of Chapters IV and V. If this orifice is in the head race the loss of head should be subtracted from the total head in order to obtain the h which really acts upon the wheel. But if the regulating gates are a part of the wheel itself, as is the case in a turbine, the loss of head should not be subtracted, because it is properly chargeable to the construction of the wheel, and not to the arrangements which furnish the supply of water. In any event that head h should be determined which is to be used in the subsequent discussions: if the efficiency of the fall is desired, the total available head is required; if the efficiency of the motor, that effective head is to be found which acts directly upon it (Art. 120).

Prob. 148. A pressure gauge at the entrance of a nozzle registers 116 pounds per square inch, and the coefficient of velocity of the nozzle is 0.98. Compute the effective velocity-head of the issuing jet.

ARTICLE 123. MEASUREMENT OF EFFECTIVE POWER.

The effective work and horse-power delivered by a water-wheel or hydraulic motor is often required to be measured. Water-power may be sold by means of the weight W, or quantity q, furnished under a certain head, leaving the consumer to provide his own motor; or it may be sold directly by the number of horse-power. In either case tests must be made from time to time in order to insure that the quantity contracted for is actually delivered and is not exceeded. It is also frequently required to measure effective work in order to ascertain the power and efficiency of the motor, either because the party who buys it has bargained for a certain power and efficiency, or because it is desirable to know exactly what the motor is doing in order to improve if possible its performance.

The effective work of a motor might be measured if it could be used to operate a pump in which are no losses of any kind. This pump might raise the same water that drives the motor through a vertical height h_1; then the effective work per second would be Wh_1, and the efficiency of the motor would be

$$e = \frac{k}{K} = \frac{Wh_1}{Wh} = \frac{h_1}{h}.$$

It is needless, however, to say that such a pump is purely imaginary.

A method in which the effective work of a small motor may be measured is to compel it to exert all its power in lifting a weight. For this purpose the weight may be attached to a cord which is fastened to the horizontal axis of the motor, and around which it winds as the shaft revolves. The wheel then

expends all its power in lifting this weight W_1 through the height h_1 in t_1 seconds, and the work performed per second then is

$$k = \frac{W_1 h_1}{t_1}.$$

This method, although practicable, is usually cumbersome in actual use, on account of the difficulty of determining t_1 with precision, since the height h_1 which can be secured is generally small.

The usual way of measuring the effective power is by means of the friction brake or power dynamometer, which is described in the next article. By this method the effective work per second k is readily determined, and then the power of the motor is

$$hp = \frac{k}{550} = 0.001818k,$$

and its efficiency is found by dividing this by the theoretic power.

The test of a hydraulic motor has for its object: first, the determination of the effective energy and power; secondly, the determination of its efficiency; and third, the determination of the speed which gives the greatest power and efficiency. If the wheel be still, there is no power; if it be revolving very fast, the water is flowing through it so as to change but little of its energy into work: and in all cases there is found a certain speed which gives the maximum power and efficiency. To execute these tests, it is not at all necessary to know how the motor is constructed or the principle of its action, although such knowledge is very valuable, and is in fact indispensable, in order to enable the engineer to suggest methods by which its operation may be improved.

Prob. 149. What is the horse-power of a motor which in 75.5 seconds lifts a weight of 320 pounds through a vertical height of 42 feet?

ARTICLE 124. THE FRICTION BRAKE, OR POWER DYNAMOMETER.

The effective work k performed by a hydraulic motor is measured by an apparatus, invented by PRONY, called the friction brake. In Fig. 80 is illustrated a simple method of applying it to a vertical shaft, the upper diagram being a plan and the lower an elevation. Upon the vertical shaft is a fixed pulley, and against this are seen two rectangular pieces of wood hollowed so as to fit it, and connected by two bolts. By turning the nuts on these bolts while the pulley is revolving, the friction can be increased at pleasure, even so as to stop the motion; around these bolts between the blocks are two spiral springs (not shown in the diagram) which press the blocks outward when the nuts are loosened. To one of these blocks is attached a cord which runs horizontally to a small movable pulley over which it passes, and supports a scale pan in which weights are placed. This cord runs in a direction opposite to the motion of the shaft, so that when the brake is tightened it is prevented from revolving by the tension caused by the weights. The direction of the cord in the horizontal plane must be such that the perpendicular let fall upon it from the centre of the shaft, or its lever arm, is constant; this can be effected by keeping the small pointer on the brake at a fixed mark established for that purpose.

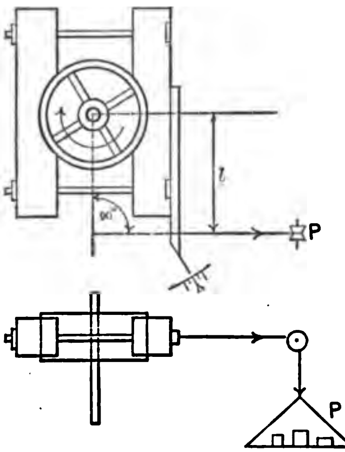

FIG. 80.

To measure the power of the wheel, the shaft is disconnected from the machinery which it usually runs, and allowed to

revolve, transforming all its work into heat by the friction between the revolving pulley and the brake which is kept stationary by tightening the nuts, and at the same time placing sufficient weights in the scale pan to hold the pointer at the fixed mark. Let n be the number of revolutions per second as determined by a counter attached to the shaft, P the tension in the cord which is equal to the weight of the scale pan and its loads, l the lever arm of this tension with respect to the centre of the shaft, r the radius of the pulley, and F the total force of friction between the pulley and the brake. Now in one revolution the force F is overcome through the distance $2\pi r$, and in n revolutions through the distance $2\pi rn$. Hence the effective work done by the wheel in one second is

$$k = F \, . \, 2\pi rn = 2\pi n \, . \, Fr.$$

The force F acting with the lever-arm r is exactly balanced by the force P acting with the lever-arm l; accordingly,

$$Fr = Pl;$$

and hence the effective work per second is

$$k = 2\pi nPl,$$

and the effective horse-power is

$$hp = \frac{2\pi nPl}{550} = 0.01142nPl. \quad . \quad . \quad . \quad . \quad (81)$$

As the number of revolutions in one second cannot be accurately read, it is usual to record the counter readings every minute or half minute; if N be the number of revolutions per minute,

$$hp = \frac{2\pi NPl}{33\,000} = 0.0001904NPl. \quad . \quad . \quad . \quad (81)'$$

It is seen that this method is independent of the radius of the pulley, which may be of any convenient size; for a small motor the brake may be clamped directly upon the shaft, but for a large one a pulley of considerable size is needed, and a special arrangement of levers is used instead of a cord.

The efficiency of the motor is now found by dividing the

effective work by the theoretic energy, or the effective power by the theoretic power; thus:

$$e = \frac{k}{K} = \frac{hp}{HP} = 6.283 \frac{nPl}{Wh}. \quad . \quad . \quad . \quad . \quad (82)$$

This same formula applies if the number of revolutions be per minute, provided that W be the weight of water which flows through the wheel per minute.

The power measured by the friction brake is that delivered at the circumference of the pulley, and does not include that power which is required to overcome the friction of the shaft upon its bearings. The shaft or axis of every water wheel must have at least two bearings, the friction of which consumes probably about 2 or 3 per cent of the power. The hydraulic efficiency of the wheel, regarded as a user of water, is hence 2 or 3 per cent greater than the value of e as given by (82).

There are in use various forms and varieties of the friction brake, but they all act upon the principle and in the manner above described. For large wheels they are made of iron, and almost completely encircle the pulley; while a special arrangement of levers is used to lift the large weight P.* If the work transformed into friction be large, both the brake and the pulley may become very hot, to prevent which a stream of cool water is allowed to flow upon them. To insure steadiness of motion it is well that the surface of the pulley should be lubricated, which for a wooden brake is well done by the use of soap.

Prob. 150. Find the power and efficiency of a motor when the theoretic energy is 1.38 horse-power, which makes 670 revolutions per minute, the weight on the brake being 2 pounds 14 ounces and its lever arm 1.33 feet.

* A paper by THURSTON in Transactions of American Society of Mechanical Engineers 1886, vol. viii., gives detailed descriptions and illustrations of the testing apparatus at Holyoke, Mass.

ARTICLE 125. TEST OF A SMALL MOTOR.

The following description and notes of a test of the 6-inch Eureka turbine in the hydraulic laboratory of Lehigh University may serve to exemplify the preceding method of determining the effective power and efficiency. The water was delivered over a weir from which it ran into a vertical penstock 15.98 square feet in horizontal cross-section. This plan of having the weir above the wheel is in general not a good one, since it is then difficult to maintain a constant head in the penstock; and it was adopted in this case on account of the lack of room below the wheel, and for other reasons which need not here be explained, as they are not related to the question in hand. The weir is briefly described in Art. 52, and the depths on its crest were determined by a hook gauge reading to thousandths of a foot. When a constant head is maintained in the penstock the quantity of water flowing through the wheel is the same as that passing the weir; if, however, the head in the penstock falls x feet per minute, the flow Q through the wheel in cubic feet per minute is

$$Q = 60q + 15.95x,$$

in which q is the flow per second through the weir, as computed by the methods of Chapter V. As the supply of water was very limited the wheel could not be run to its full capacity. There was no leakage from the penstock, and the slight leakage through the gate of the turbine is properly included in the value of q, since it assists in running the wheel.

The level of water in the penstock above the wheel was read upon a head gauge consisting of a glass tube behind which a graduated scale was fixed, the zero of which was a little above the water level in the tail race. The latter level was read upon a fixed graduated scale having its zero in the same horizontal plane as the first; these readings were hence essentially nega-

tive. The head upon the wheel is then found by adding the readings of the two gauges.

The vertical shaft of the turbine, being about 15 feet long, was supported by four bearings, and to a small pulley upon its upper end was attached the friction dynamometer, as described in the last article. The number of revolutions was read from a counter placed in the top of this shaft. The observations were taken at minute intervals, electric bells giving the signals, so that precisely at the beginning of each minute simultaneous readings were taken by observers at the weir, at the head gauge, at the tail gauge, and at the counter, the operator at the brake continually keeping it in equilibrium with the resisting weight in the scale pan by slightly tightening and loosening the nuts as required. The following table gives the notes of four sets,

TABLE XXIII. TEST OF A 6-INCH EUREKA TURBINE.

Time on April 13, 1888.	Depth on Weir Crest. Feet.	Penstock Gauge. Feet.	Tail-race Gauge. Feet.	Revolutions in One Minute.	Weight on Brake. Pounds.	Remarks.
3^h 17^m	0.288	11.25	−0.21	635	2.5	
18	0.289	11.17	0.20	625	"	Length of weir, $b = 1.909$ feet.
19	0.289	11.13	0.21	635	"	
20	0.288	11.10	0.21		"	Length of lever-arm on brake, $l = 1.431$ feet.
3^h 22^m	0.287	10.81	−0.20	535	3.0	
23	0.287	10.69	0.20	540	"	Gate of wheel $\frac{3}{4}$ths open during all experiments.
24	0.287	10.62	0.21	535	"	
25	0.286	10.57	0.21		"	
3^h 27^m	0.288	10.64	−0.23	600	2.5	Temperature of the water not taken.
28	0.288	10.72	0.22	600	"	
29	0.291	10.80	0.21	615	"	
30	0.290	10.90	0.20		"	
3^h 32^m	0.290	10.72	−0.20	445	3.5	
33	0.291	10.69	0.20	440	"	
34	0.291	10.66	0.20	440	"	
35	0.292	10.64	0.20		"	

each lasting three minutes, the weight in the scale pan being different in each. In the intervals between the sets the wheel was kept running and observations were regularly taken; but they are not used, owing to the disturbance of the permanent motion consequent upon changing the weight in the scale-pan.

The following table gives the results of the computations made from the above notes for each minute interval. The second column is derived from formula (33) of Art. 53, using

TABLE XXIV. RESULTS OF TEST OF A 6-INCH TURBINE.

Interval of Time.	Discharge over Weir. Cub. Feet per Minute.	Fall in Penstock. Feet.	Flow through Wheel. Cub. Feet per Minute.	Head on Wheel. Feet.	Theoretic Horse-power of the Water.	Effective Horse-power of the Wheel.	Efficiency of the Wheel. Per Cent.
17^m to 18^m	58.49	+0.08	59.77	11.41	1.290	0.433	33.6
18 to 19	58.66	+0.04	59.30	11.36	1.274	0.426	33.4
19 to 20	58.49	+0.03	58.97	11.32	1.262	0.433	34.3
22^m to 23^m	58.05	+0.13	60.13	10.95	1.245	0.437	35.1
23 to 24	58.05	+0.07	59.17	10.86	1.215	0.441	36.3
24 to 25	57.88	+0.05	58.68	10.80	1.198	0.437	36.5
27^m to 28^m	58.36	−0.08	57.08	10.91	1.175	0.409	34.7
28 to 29	58.51	−0.08	57.23	10.97	1.187	0.409	34.4
29 to 30	59.10	−0.10	57.50	11.06	1.203	0.419	34.8
32^m to 33^m	59.10	+0.03	59.58	10.90	1.228	0.424	34.5
33 to 34	59.26	+0.03	59.74	10.87	1.228	0.420	34.2
34 to 35	59.41	+0.02	59.73	10.85	1.226	0.420	34.3

the coefficient corresponding to the given length of weir and depth on crest. The third column is obtained by taking the differences of the observed readings of the penstock head gauge. The fourth column is the value of Q found as above explained. The fifth column is the mean head h on the wheel during the minute, as derived from the observed readings of head and tail gauge. The sixth column is found by formula (80), using for W its value $\frac{1}{60}wQ$, in which w is taken at 62.4

pounds per cubic foot. The seventh column is computed from formula (81)′; and the last column is found by dividing the numbers in the seventh by those in the sixth column.

These results show that the mean efficiency of the wheel and shaft under the conditions stated was about 35 per cent; also, that the efficiency in the second set is the highest and that in the first is the lowest. The following recapitulation of the means for each set show that the reason for the variation in efficiency is the variation in speed, and it is to be concluded

	N.	h.	Q.	e.
1st set,	632	11.36	59.31	33.8
3d set,	605	10.98	57.27	34.6
2d set,	536	10.87	59.33	36.0
4th set,	441	10.87	59.68	34.3

that with a head of about 11 feet this wheel at three-fourths gate has a maximum efficiency of 36.0 per cent when running at about 535 revolutions per minute. If four points be plotted, taking the values of N as abscissas and those of e as ordinates, and a curve be drawn through them, it will be seen that quite material variations in the speed may occur without sensibly affecting the efficiency; thus N may range from 475 to 575 revolutions per minute without making e lower than 0.35.

Prob. 151. Compute, using four-figure logarithms, the results in the last three columns of the above table.

ARTICLE 126. LOWELL AND HOLYOKE TESTS.

The work of FRANCIS on the experiments made by him at Lowell will always be a classic in American hydraulic literature, for the methods therein developed for measuring the theoretic power of a water-fall, and the effective power utilized by the wheel, are models of careful and precise experimentation.* In determining the speed of the wheel he used a method

* Lowell Hydraulic Experiments, 1st Edition, 1855; 4th, 1883.

somewhat different from that above explained, namely, the counter attached to the shaft was connected with a bell which struck at the completion of every 50 revolutions; the observer at the counter had then only to keep his eye upon the watch, and to note the time at certain designated intervals—say at every sixth stroke of the bell. The number of revolutions per second was then obtained by dividing the number of revolutions in the interval by the number of seconds, as determined by the watch. This method requires a stop-watch in order to do good work, unless the observer be very experienced, as an error of one second in an interval of one minute amounts to 1.7 per cent.

The Holyoke Water Power Company has a permanent flume for testing turbines arranged with a weir which can be varied up to lengths of 20 feet, so as to test the largest wheels which are constructed. As the expense of fitting up the apparatus for testing a large turbine at the place where it is to be used is often great, it is sometimes required in contracts that the wheel shall be sent to Holyoke to be tested, that being the only place in the United States where a special outfit for such work exists. The wheel is mounted in the testing flume, and there, by the methods explained in the preceding articles, it is run at different speeds in order to determine the speed which gives the maximum efficiency as well as the effective power developed at each speed. As the efficiency of a turbine varies greatly with the position of the gate which admits the water to it, tests are made with the gate fully opened and at various partial openings. The results thus obtained are not only valuable in furnishing full information concerning the effective power and efficiency of the wheel, but they also convert the turbine into a water meter, so that when running under the same head as during the tests the quantity of water which passes through it can at any time be approximately ascertained.

The following table gives a report of the Holyoke Water Company of the test of an 80-inch outward-flow BOYDEN turbine made September 22, 1885.* The headings of the several

TABLE XXV. TEST OF AN 80-INCH BOYDEN TURBINE.

Number of the Experiment.	Proportional Part of the full Opening of the Speed-gate.	Proportional Part of the full Discharge of the Wheel; being the Discharge at full Gate when giving best Efficiency	Head acting on the Wheel. Feet.	Duration of the Experiment. Min.	Revolutions of the Wheel. Per Min.	Quantity of Water discharged by the Wheel. Cub. Ft. per Second.	Power developed by the Wheel. h. p.	Efficiency of the Wheel. Per Cent
21	1.000	0.992	17.16	5	63.50	117.01	172.57	75.85
20	"	1.000	17.27	5	70.00	118.37	177.41	76.60
19	"	1.008	17.33	3	75.00	119.53	178.63	76.11
18	"	1.020	17.34	3	80.00	121.15	178.32	74.92
17	"	1.036	17.21	2	86.00	122.41	178.57	74.81
16	"	1.056	17.21	5	93.20	124.74	176.44	72.54
15	"	1.082	17.19	3	100.00	127.73	167.94	67.51
14	0.753	0.923	17.26	4	65.00	109.22	148.86	69.69
13	"	0.931	17.35	4	71.00	110.42	151.76	69.91
12	"	0.944	17.33	3	77.17	111.94	153.16	69.68
11	"	0.957	17.34	3	82.83	113.52	151.75	68.04
10	"	0.986	17.34	3	93.33	116.98	151.04	65.72
9	"	0.999	17.27	4	97.75	118.24	140.29	60.63
8	"	1.018	17.23	4	104.50	120.36	130.83	55.68
33	0.609	0.849	17.64	4	65.00	101.60	130.99	64.51
32	"	0.861	17.57	3	71.00	102.78	130.08	63.58
31	"	0.876	17.53	4	78.00	104.45	128.61	62.00
30	"	0.892	17.45	4	84.75	106.19	124.22	59.16
7	0.438	0.706	17.68	3	64.00	84.56	84.03	49.61
6	"	0.716	17.69	4	69.25	85.74	82.47	47.99
5	"	0.723	17.69	4	74.75	86.57	79.89	46.04
4	"	0.731	17.66	4	79.87	87.54	75.60	43.16
3	"	0.746	17.62	4	86.50	89.23	68.67	38.55
2	"	0.762	17.64	3	94.33	91.12	57.61	31.63
1	"	0.773	17.61	4	100.50	92.36	46.03	24.98
27	0.310	0.555	18.03	4	61.75	67.11	43.37	31.63
26	"	0.570	18.01	4	69.25	68.87	40.18	28.59
25	"	0.584	18.02	3	77.00	70.59	35.27	24.47
28	"	0.597	18.23	3	85.00	72.62	28.55	19.03
29	"	0.608	18.13	3	90.67	73.74	19.38	12.79
24	0.200	0.401	18.17	3	54.33	48.68	18.25	18.21
23	"	0.409	18.11	3	61.67	49.52	15.06	14.83
22	"	0.413	18.07	4	66.50	50.03	12.18	11.89

* Kindly furnished by Mr. CLEMENS HERSCHEL, Hydraulic Engineer of the Holyoke Water Power Company.

columns sufficiently designate their meaning, and only a few words need be added explanatory of the method by which they were obtained. The numbers in the second column were found by actual measurement of the clear space beneath the gate, the space at full gate being called unity; those in the fourth column are derived from the head and tail race gauges; those in the sixth column by dividing the total number of revolutions during the experiment by its length in minutes; those in the seventh by the measurement of the water over the weir; those in the eighth from the friction dynamometer by the use of formula (81)′; and those in the last column were computed by (82). The quantities in the third column result from the division of those in the seventh by 118.37, that being the discharge at full gate for maximum efficiency; it is seen from these that the discharge depends not only upon the head, but on the velocity of the wheel, and that it always increases when the speed increases.*

The following data regarding the dimensions of this wheel are here also noted, as it may be necessary to refer to it again when the subject of turbines is discussed:

Outer radius of wheel	$r_1 = 3.3167$ feet;
Inner radius of wheel	$r = 2.6630$ feet;
Outer radius of guide case	$r_0 = 2.5911$ feet;
Outer depth of buckets	$d_1 = 0.722$ feet;
Inner depth of buckets	$d = 0.741$ feet;
Outer area of buckets	$a_1 = 4.61$ square feet;
Inner area of buckets	$a = 12.12$ square feet;
Outer area of guide orifices	$a_0 = 4.76$ square feet;
Exit angle of buckets	$\beta = 13.5$ degrees;
Entrance angle of buckets	$\phi = 90$ degrees;
Entrance angle of guides	$\alpha = 24$ degrees;
Number of buckets, 52.	Number of guides, 32.

* For further examples of tests at Holyoke see a paper by THURSTON in Transactions American Society Mechanical Engineers, vol. viii. p. 359.

Prob. 152. In experiment 32 on the outward-flow turbine only about 12 per cent of the theoretic power is utilized. How is the remaining 88 per cent expended?

ARTICLE 127. WATER POWER.

In 1880 there was employed in the United States a total of 3 410 837 horse-power, of which about 36 per cent was derived from water and about 64 per cent from steam. It has been estimated that the rivers of the United States can furnish about 200 000 000 horse-powers, so that the possibilities for the future are almost unlimited, and when coal becomes high in price water is sure to take the place of steam.

Water-power is often sold by what is called the "mill power," which may be roughly supposed to be such a quantity as the average mill requires, but which in any particular case must be defined by a certain quantity of water under a given head. Thus at Lowell the mill power is 30 cubic feet per second under a head of 25 feet, which is equivalent to 85.2 theoretic horse-power. At Minneapolis it is 30 cubic feet per second under 22 feet head, or 75 theoretic horse-power. At Holyoke it is 38 cubic feet per second under 20 feet head, or 86.4 theoretic horse-power. This seems an excellent way to measure power when it is to be sold or rented, as the head in any particular instance is not subject to much variation; or if so liable, arrangements must be adopted for keeping it nearly constant, in order that the machinery in the mill may be run at a tolerably uniform rate of speed. Thus nothing remains for the water company to measure except the water used by the consumer. The latter furnishes his own motor, and is hence interested in securing one of high efficiency, that he may derive the greatest power from the water for which he pays. The perfection of American turbines is undoubtedly largely due to this method of selling power, and the consequent desire of the

mill owners to limit their expenditure of water. The turbine itself when tested and rated becomes a meter by which the company can at any time determine the quantity of water that passes through it. At Holyoke the cost of one mill power for 16 hours a day is $300 per annum.*

The available power of natural water-falls is very great, but it is probably exceeded by that which can be derived from the tides and waves of the ocean. Twice every day, under the attraction of the sun and moon, an immense weight of water is lifted, and it is theoretically possible to derive from this a power due to its weight and lift. Continually along every ocean beach the waves dash in roar and foam, and energy is wasted in heat which by some device might be utilized in power. The expense of deriving power from these sources is indeed greater than that of the water wheel under a natural fall, but the time may come when the profit will exceed the expense, and then it will certainly be done. Coal and wood and oil may become exhausted, but as long as the force of gravitation exists, and the ocean remains upon which it can act, heat, light, and power can be generated in quantity practically without limit.

Prob. 153. Show that over 600 horse-power is wasted in heat for every square mile of ocean surface where the rise and fall of the tide is 3 feet.

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* BRECKENRIDGE, Journal of Engineering Society of Lehigh University, 1887, vol. ii. p. 34; an article giving a detailed account of the water power at Holyoke.

CHAPTER XI.

DYNAMIC PRESSURE OF FLOWING WATER.

ARTICLE 128. DEFINITIONS AND PRINCIPLES.

The pressures exerted by moving water against surfaces which change its direction or check its velocity are called dynamic, and they follow very different laws from those which govern the static pressures that have been discussed and used in the preceding chapters. A static pressure due to a certain head may cause a jet to issue from an orifice; but this jet in impinging upon a surface may cause a dynamic pressure less than, equal to, or greater than that due to the head. A static pressure at a given point in a mass of water is exerted with equal intensity in all directions; but a dynamic pressure is exerted in different directions with different intensities. In the following chapters the words static and dynamic will generally be prefixed to the word pressure, so that no intellectual confusion may result.

The dynamic pressure exerted by a stream flowing with a given velocity against a surface at rest is evidently equal to that produced when the surface moves in still water with the same velocity. This principle was applied in Art. 109 in rating the current meter, whose vanes move under the impulse of the impinging water. The dynamic pressure exerted upon a body by a flowing stream hence depends upon the velocity of the stream and surface.

The impulse of a jet or stream of water is the dynamic pressure which it is capable of producing in the direction of its motion when its velocity is entirely destroyed in that direction.

This can be done by deflecting the jet normally sidewise by a fixed surface; if the surface is smooth, so that no energy is lost in frictional resistances, the actual velocity remains unaltered, but the velocity in the original direction has been rendered null. In Art. 32 it is proved that the theoretic force of impulse of a stream of cross-section a and velocity v is

$$F = W\frac{v}{g} = wq\frac{v}{g} = 2wa\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad (83)$$

in which W and q are the weight and volume delivered per second, and w is the weight of one cubic unit of water. This equation shows that the dynamic pressure that may be produced by impulse is equal to the static pressure due to twice the head corresponding to the velocity v. It would then be expected that if two equal orifices or tubes be placed exactly opposite, as in Fig. 81, and a loose plate be placed vertically against one of them, that the dynamic pressure upon the plate caused by the impulse of the jet issuing from A under the head h would balance the static pressure caused by the head $2h$. This conclusion has been confirmed by experiment, when the tube A has a smooth inner surface and rounded inner edges so that its coefficient of discharge is unity.

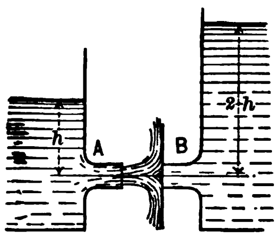

FIG. 81.

The reaction of a jet or stream is the backward dynamic pressure, in the line of its motion, which is exerted against a vessel out of which it issues, or against a surface away from which it moves. This is equal and opposite to the impulse, and the equation above given expresses its value and the laws which govern it.

The expression for the reaction or impulse F given by (83) may be also proved as follows: The definition by which forces

are compared with each other is, that forces are proportional to the accelerations which they can produce. The weight W if allowed to fall acquires the acceleration g; the force F which can produce the acceleration v is hence related to W and g by the equation $F \div v = W \div g$.

Impulse and reaction in a cross-section of a stream flowing with constant velocity and direction are forces which can be exerted, and hence like energy are potential. If the direction of the stream be changed by opposing obstacles, the impulse and reaction produce dynamic pressure; if in making this change the absolute velocity is retarded, energy is converted into work. Impulse and reaction are of no practical value, except in so far as the resulting dynamic pressures may be utilized for the production of work. For this purpose water is made to impinge upon moving vanes, which alter both its direction and velocity, thus producing a dynamic pressure P, which overcomes in each second an equal resisting force through the space u. The work done per second is then

$$k = Pu.$$

It is the object in designing a hydraulic motor to make this work as large as possible, and for this purpose the most advantageous values of P and u are to be selected.

The word impact, which is sometimes erroneously used to mean impulse or pressure, properly refers to those cases where energy is lost through changes of cross-section (Art. 68), or in eddies and foam, as when a jet impinges into water or upon a rough plane surface. When work is to be utilized, impact should be avoided as far as possible.

Prob. 154. If a jet is one inch in diameter, how many gallons per second must it deliver in order that its impulse may be 100 pounds?

ARTICLE 129. EXPERIMENTS ON IMPULSE AND REACTION.

In Fig. 82 is shown a simple device by which the dynamic pressure P exerted upon a surface by the impulse and reaction of a jet that glides over it can be directly weighed. It consists merely of a bent lever supported on a pivot at O, and having a plate A attached at lower end of the vertical arm upon which the stream impinges, while weights applied at the end of the other arm measure the dynamic pressure produced by the impulse. By means of an apparatus of this nature, experiments have been made by BIDONE, WEISBACH, and others, the results of which will now be stated.

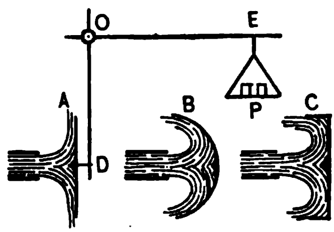

FIG. 82.

When the surface upon which the stream impinges is a plane normal to the direction of the stream, as shown at A, the dynamic pressure P closely agrees with that given by the theoretic formula for F in the last article, viz.,

$$P = W\frac{v}{g} = 2wa\frac{v^2}{2g}$$

being about 2 per cent greater according to BIDONE, and about 4 per cent less according to WEISBACH. The actual value of P was found to vary somewhat with the size of the plate, and with its distance from the end of the tube from which the jet issued.

When the surface upon which the stream impinges is curved, as at B, or so arranged that the water is turned backward from the surface, the value of the dynamic pressure P was found to be always greater than the theoretic value, and that it increased

the greater the amount of backward inclination. When a complete reversal of the original direction of the water was obtained, as at C, it was found that P, as measured by the weights, was nearly double the value of that against the plane. This is explained by stating that as long as the direction of the flow is toward the surface the dynamic pressure of its impulse is exerted upon it; when the water flows backward away from the surface the dynamic pressure of its reaction is also exerted upon it. The sum of these is

$$P = F + F = 2W\frac{v}{g} = 4wa\frac{v^2}{2g},$$

which agrees with the results experimentally obtained.

An experiment by MOROSI * shows clearly that the dynamic pressure against a surface may be increased still further by the device shown in Fig. 83, where the stream is made to perform two complete reversals upon the surface. He found that in

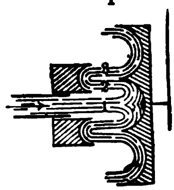

FIG. 83.

this case the value of the dynamic pressure was 3.32 times as great as that against a plane, or $P = 3.32\,F$, whereas theoretically the 3.32 should be 4. In this case, as in those preceding, the water in passing over the surface loses energy in friction and foam, so that its velocity is diminished, and it should hence be expected that the experimental values of the dynamic pressures would be less than the theoretic values, as in general they are found to be.

While the experiments here briefly described thoroughly confirm the results of theory, they further show it is the change in direction of the velocity when in contact with the surface which produces the dynamic pressure. If the stream strikes

* RUHLMAN's Hydromechanik (Hannover, 1879), p. 586.

normally against a plane, the direction of its velocity is changed 90°, and this is the same as the entire destruction of the velocity in its original direction, so that the dynamic pressure P should agree with the impulse F. This important principle of change in direction will be theoretically exemplified later.

The dynamic pressure produced by the direct reaction of water when issuing from a vessel was measured by EWART with the apparatus shown in Fig. 84, which will be readily understood without a detailed description. The discussion of these experiments made by WEISBACH* shows that the measured values of P were from 2 to 4 per cent less than the theoretic value F as given by (83), so that in this case also theory and observation are in accordance.

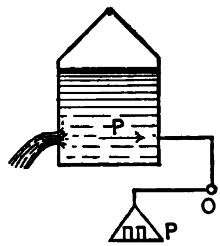

FIG. 84.

An experiment by UNWIN,† illustrated in Fig. 85, is very interesting, as it perhaps explains more clearly than formula (83) why it is that the dynamic pressure due to impulse is double the static pressure. Two vessels having converging tubes of equal size were placed so that the jet from A was directed exactly into B. The head in A was kept uniform at $20\frac{1}{2}$ inches, when it was found that the water in B continued to rise until a head of 18 inches was reached. All the water admitted into A was thus lifted in B by the impulse of the jet, with a loss of $2\frac{1}{2}$ inches of head, which was caused by foam and friction. If such losses could be entirely avoided, the water in B might be raised to the same level as that in A. In the case shown in

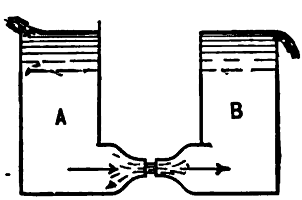

FIG. 85.

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* Theoretical Mechanics, COXE's translation, p. 1004.

† Encyclopædia Britannica, 9th Edition, vol. xii. p. 467.

the figure where the water overflows from *B*, the impulse of the jet has not only to overcome the static pressure due to the head *h*, but also to furnish the dynamic pressure equivalent to a second head *h* in order to raise the water through that height. But the level in *B* can never rise higher than in *A*, for the velocity-head of the jet cannot be greater than that of the static head which generates it.

Prob. 155. In Fig. 82 the diameter of the tube is 1 inch, and it delivers 0.3 cubic feet per second. Compute the theoretic dynamic pressure against the plane.

ARTICLE 130. SURFACES AT REST.

Let a jet of water whose cross-section is *a* impinge in permanent flow with the uniform velocity *v* upon a surface at rest. Let the surface be smooth, so that no resisting forces of friction exist, and let the stream be prevented from spreading laterally. The water then passes over the surface, and leaves it

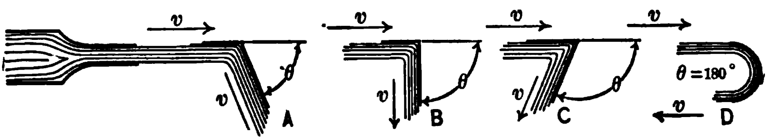

FIG. 86.

with the original velocity *v*, producing upon it a dynamic pressure whose value depends upon its change of direction. At *B* in Fig. 86 the stream is deflected normal to its original direction, and at *D* a complete reversal is effected, Let θ be the angle between the initial and final directions, as shown. It is required to determine the dynamic pressure exerted upon the surface in the same direction as that of the jet. In Fig. 86, as in those that follow, the stream is supposed to lie in a horizontal plane, so that no acceleration or retardation of its velocity will be produced by gravity.

The stream entering upon the surface exerts its impulse F in the same direction as that of its motion; leaving the surface it exerts its reaction F in opposite direction to that of its motion. Let P be the dynamic pressure thus produced in the direction of the initial motion, F_1 the component of the reaction F in the same direction. Then, if θ be less than 90°,

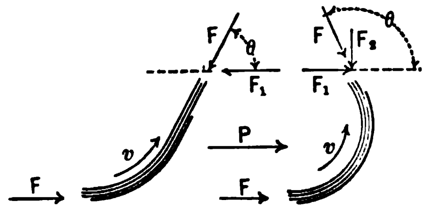

FIG. 87.

$$P = F - F_1 = F(1 - \cos\theta);$$

and if θ be greater than 90°,

$$P = F + F_1 = F + F\cos(180° - \theta) = F(1 - \cos\theta).$$

Both cases thus give the same result, and inserting for F its value as given by (83);

$$P = (1 - \cos\theta)W\frac{v}{g}, \quad . \quad . \quad . \quad . \quad . \quad (84)$$

which is the formula for the dynamic pressure in the direction of the impinging jet. If in this $\theta = 0°$, the stream glides along the surface without changing its direction, and P becomes zero; if θ is 90°, the dynamic pressure is

$$P = F = W\frac{v}{g};$$

and if θ becomes 180° a complete reversal of direction is obtained, and

$$P = 2F = 2W\frac{v}{g}.$$

These theoretic conclusions agree with the experimental results described in the last article.

The resultant dynamic pressure exerted upon the surface is found by combining by the parallelogram of forces the impulse

F and the equal reaction F. In Fig. 87 it is seen that this resultant bisects the angle $180 - \theta$, and that its value is

$$P' = 2F \cos \tfrac{1}{2}(180 - \theta) = 2 \sin \tfrac{1}{2} \theta . W\frac{v}{g}.$$

It is usually, however, more important to ascertain the pressure in a given direction than the resultant. This can be found by taking the component of the resultant in that direction, or by taking the algebraic sum of the components of the initial impulse and the final reaction.

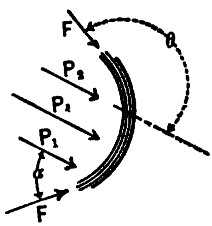

FIG. 88.

To find the dynamic pressure P in a direction which makes an angle α with the entering and the angle θ with the departing stream, the components in that direction are

$$P_1 = F \cos \alpha, \qquad P_2 = F \cos \theta;$$

and the algebraic sum of these is

$$P = F(\cos \alpha - \cos \theta) = (\cos \alpha - \cos \theta) W\frac{v}{g}. \quad . \quad (84)'$$

This becomes equal to F when $\alpha = 0$ and $\theta = 90°$, as at B in Fig. 86, and also when $\alpha = 90°$ and $\theta = 180°$. When $\alpha = 0°$ and $\theta = 180°$ the entering and departing streams are parallel, as at D in Fig. 86, so that the value of P is $2F$, which in this case is the same as the resultant pressure.

The formulas here deduced are entirely independent of the form of the surface, and of the angle with which the jet enters upon it. It is clear, however, if, as in the planes in Fig. 86, this angle is such as to allow shock to occur, that foam and changes in cross-section may result which will cause energy to be dissipated in heat. These losses will diminish the velocity v and decrease the theoretic dynamic pressure. These effects cannot be formulated, but it is a general principle, which is confirmed

by experiment, that they may be largely avoided by allowing the jet to impinge tangentially upon the surface.

In all the foregoing formulas the weight W which impinges upon the surface per second is the same as that which issues from the orifice or nozzle that supplies the stream, and its value is

$$W = wq = wav.$$

To find W it is hence necessary to determine the discharge q by the methods explained in the preceding chapters, or to measure a, the area of the cross-section of the stream, and to ascertain by some method the mean velocity v.

Prob. 156. If F is 10 pounds, $\alpha = 0°$, and $\theta = 60°$, show that the pressure parallel to the direction of the jet is 5 pounds, that the pressure normal to that direction is 8.66 pounds, and that the resultant dynamic pressure is 10 pounds.

ARTICLE 131. CURVED PIPES AND CHANNELS.

The dynamic pressures discussed in the preceding article have been those caused by jets, or isolated streams, of water. There is now to be considered the case of dynamic pressures caused by streams flowing in pipes, conduits, or channels of any kind; these are sometimes called limited or bounded streams, the boundary being the surface whose cross-section is the wetted perimeter. When such a stream is straight and of uniform section, and all its filaments move with the same velocity v, the impulse, or the pressure which it can produce, is the quantity F given by the general expression in Art. 128; under these conditions it exerts no dynamic pressure, but if a body be immersed and held stationary, pressure is produced upon it. If its direction changes in an elbow or bend, pressure upon the

bounding surface is produced; if its cross-section increases or decreases, pressure is developed or absorbed.

The resultant dynamic pressure P' upon a curved pipe which runs full of water with the uniform velocity v depends upon the angle θ between the initial and final directions, and

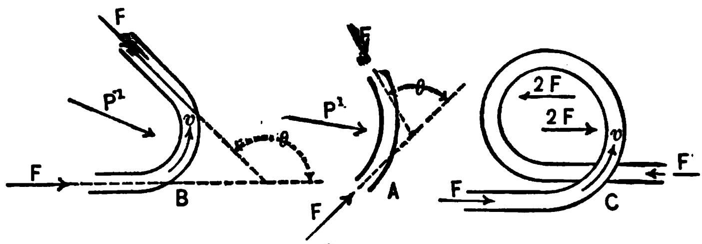

FIG. 89.

must be the same as that produced upon a surface by an impinging jet which passes over it without change in velocity. The formula of Art. 130 then directly applies, and

$$P' = 2 \sin \tfrac{1}{2}\theta \, . \, F = 2 \sin \tfrac{1}{2}\theta \, . \, W\frac{v}{g}.$$

If $\theta = 0°$, there is no bend, and $P' = 0$; if $\theta = 180°$, the direction of flow is reversed, and $P' = 2F$. If the direction is twice reversed, as at C in Fig. 89, the value of θ is 360°, and the resultant is the initial impulse F minus the final reaction F, or simply zero; in this case, however, there may be a couple which tends to twist the pipe, unless the impulse and reaction lie in the same line.

The total dynamic pressure exerted upon the curved pipe may be found by taking the sum of all the elementary radial pressures. For this purpose let the pipe at A in Fig. 89 have the length δl and let θ be nearly 0°, so that its value is the elementary angle $\delta\theta$. Then in the above formula P' becomes the elementary radial pressure δP_1, and

$$\delta P_1 = 2 \sin \tfrac{1}{2}\delta\theta \, . \, F = F\delta\theta.$$

Now for a circular curve whose radius is R, the value of δl is $R\delta\theta$; and accordingly the elementary radial pressure for that case is expressed by the dfferential equation

$$\delta P_1 = \frac{F}{R}\delta l.$$

The total radial pressure P_1 upon a circular curve whose length is l is the integral of this equation between the limits 0 and P_1 for the first member and 0 and l for the second, or

$$P_1 = \frac{Fl}{R} = 2wa\,\frac{l}{R}\,\frac{v^2}{2g}.$$

This dynamic pressure does no work and offers no direct resistance in the direction of the flow; but in being transmitted through the water to the outer side of the pipe it causes cross-currents which consume energy. This expression for radial pressure is the same as that given by the theory of centrifugal force. It is not strictly exact unless all the filaments have the same velocity v, which in a curved pipe is probably never the case.

The same reasoning applies approximately to the curves of conduits, canals, and rivers. In any length l there exists a radial dynamic pressure P_1, acting toward the outer bank and causing currents in that direction, which, in connection with the greater velocity that naturally there exists, tends to deepen the channel on that side. This may help to explain the reason why rivers run in winding courses. At first the curve

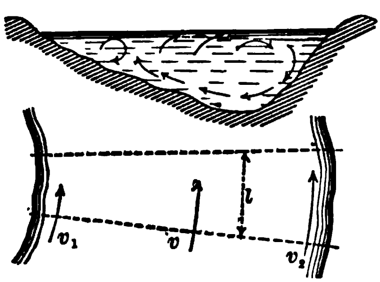

FIG. 90.

may be slight, but the radial flow due to the dynamic pres-

sure causes the outer bank to scour away; this increases the velocity v_2 and decreases v_1 (Fig. 90), and this in turn hastens the scour on the outer and allows material to be deposited on the inner side. Thus the process continues until a state of permanency is reached, and then the existing forces tend to maintain the curve. The cross-currents which the radial pressure produces have been actually seen in experiments devised by THOMSON,* and it is found that they move in the manner shown in Fig. 90, the motion toward the outer bank being in the upper part of the section, while along the wetted perimeter they flow toward the inner bank.

The elevation of the water on the outer bank of a bend is higher than on the inner. This is only a practical consequence of the radial dynamic pressure, as in straight streams it is also seen that the water surface is curved, the highest elevation being where the velocity is greatest. In this case cross-currents are also observed which move near the surface toward the centre of the stream, and near the bottom toward the banks, their motion being due to the disturbance of the static pressure consequent upon the varying water level.

Prob. 157. Why is it that streams of slight slope have the most winding courses?

ARTICLE 132. IMMERSED BODIES.

When a body is immersed in a flowing stream, or when it is moved in still water, so that filaments are caused to change their direction, a dynamic pressure is exerted or overcome. The theoretic determination of the intensity of this pressure is difficult, if not impossible, and will not be here attempted;

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* Proceedings Royal Society, 1877, p. 356.

in fact, experiment alone can furnish reliable conclusions. It is, however, to be inferred from what has preceded, that the dynamic pressure

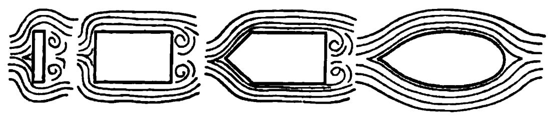

FIG. 91.

in the direction of the motion is proportional to the force of impulse of a stream whose cross-section is the same as that of the body, or

$$P = m \, . \, wa \frac{v^2}{2g},$$

in which m is a number depending upon the length and shape of the immersed portion, and whose value is 2 for a jet impinging normally upon a plane.

Experiments made upon small plates held normally to the direction of the flow show that the value of m lies between 1.25 and 1.75, the best determinations being near 1.4 and 1.5. It is to be expected that the dynamic pressure on a plate in a stream would be less than that due to the impulse of a jet of the same cross-section, as the filaments of water near the outer edges are crowded sideways, and hence do not impinge with full normal effect, and the above results confirm this supposition. The few experiments on record were made with small plates, mostly less than 2 square feet area, and they seem to indicate that m is greater for large surfaces than for small ones.

The determination of the dynamic pressure upon the end of a cylinder, as at B in Fig. 91, is difficult because of the resisting friction of the sides; but it is well ascertained to be less than that upon a plane of the same area, and within certain limits to decrease with the length. For a conical or wedge-shaped body the dynamic pressure is less than that upon the cylinder, and it is found that its intensity is much modified by the shape of the rear surface.

When a body is so formed as to gradually deflect the filaments of water in front, and to allow them to gradually close in again upon the rear, the impulse of the front filaments upon the body is balanced by the reaction of those in the rear, so that the resultant dynamic pressure is zero. The forms of boats and ships should be made so as to secure this result, and then the propelling force has only to overcome the frictional resistance of the surface upon the water.

The dynamic pressure produced by the impulse of ocean waves striking upon piers or lighthouses is often very great. The experiments of STEVENSON* on Skerryvore Island, where the waves probably acted with greater force than usual, showed that during the summer months the mean dynamic pressure per square foot was about 600 pounds, and during the winter months about 2100 pounds, the maximum observed value being 6100 pounds. At the Bell Rock lighthouse the greatest value observed was about 3000 pounds per square foot. The observations were made by allowing the waves to impinge upon a circular plate about 6 inches in diameter, and the pressure produced was registered by the compression of a spring.

Prob. 158. Compute the probable dynamic pressure upon a surface one foot square when immersed in a current whose velocity is 8 feet per second, the direction of the current being normal to the surface.

ARTICLE 133. MOVING VANES.

A vane is a plane or curved surface which moves in a given direction under the dynamic pressure exerted by an impinging jet or stream. The direction of the motion of the vane depends upon the conditions of its construction; for example, the vanes of a water wheel can only move in a circumference around its axis. The simplest case for consideration, however,

* RANKINE's Civil Engineering, p. 756.

is that where the motion is in a straight line, and this alone will be considered in this article. The plane of the stream and vane is to be taken as horizontal, so that no direct action of gravity can influence the action of the jet.

Let a jet with the velocity v impinge upon a vane which moves in the same direction with the velocity u, and let the velocity of the jet relative to the surface at the point of exit make an angle β with the reverse direction of u, as shown in Fig. 92. The velocity of the stream relative to the surface is $v - u$, and the dynamic pressure is the same as if the surface were at rest, and the stream moving with the absolute velocity $v - u$. Hence formula (84) directly applies, replacing v by $v - u$, and θ by $-\beta$, and

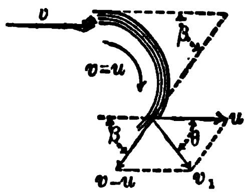

FIG. 92.

$$P = (1 + \cos\beta) W \frac{v - u}{g}.$$

In this formula W is not the weight of the water which issues from the nozzle, but that which strikes and leaves the vane, or $W = wa(v - u)$; for under the condition here supposed the vane moves continually away from the nozzle, and hence does not receive all the water which it delivers.

Another method of deducing the last equation is as follows: At the point of exit let lines be drawn representing the velocities $v - u$ and u; then completing the parallelogram, the line v_1 is the absolute velocity of the departing jet (Art. 33). Let θ be the angle which v_1 makes with the direction of u, and β as before the angle between $v - u$ and the reverse direction of u. Then the dynamic pressure is that due to the absolute impulse of the entering and departing streams; the former of these has the value $\frac{W}{g}v$ and the latter the value $\frac{W}{g}v_1 \cos\theta$.

Hence it is expressed by the formula

$$P = \frac{W}{g}(v - v_1 \cos \theta).$$

But from the triangle between v_1 and u

$$v_1 \cos \theta = u - (v - u) \cos \beta.$$

Inserting this, the value of the dynamic pressure is

$$P = \frac{W}{g}(v - u)(1 + \cos \beta),$$

which is the same as that found before. If $\beta = 180°$ there is no pressure, and if $\beta = 0°$ the stream is completely reversed, and P attains its maximum value. The dynamic pressure may be exerted with different intensities upon different parts of the vane, but its total value in the direction of the motion is that given by the formula.

Usually the direction of the motion is not the same as that of the jet. This case is shown in Fig. 93, where the arrow marked F designates the direction of the impinging jet, and that marked P the direction of the motion of the vane, α being the angle between them. The jet having the velocity v impinges upon the vane at A, and in passing over it exerts a dynamic pressure P which causes it to move with the velocity u. At A let lines be drawn representing the intensities and directions of v and u, and let the parallelogram of velocities be formed as shown; the line V then represents the velocity of the water relative to the vane at A. The stream passes over the surface and leaves it at B with the same relative velocity V, if not retarded by

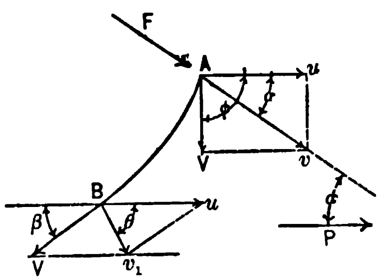

FIG. 93.

friction or foam. At B let lines be drawn to represent u and V, and let β be the angle which V makes with the reverse direction of u; let the parallelogram be completed, giving v_1 for the absolute velocity of the departing water, and let θ be the angle which it makes with u. The total pressure in the direction of the motion is now to be regarded as that caused by the components in that direction of the initial and the final impulse of the water. The impulse of the stream before striking the vane is $\frac{W}{g}v$, and its component in the direction of the motion is $\frac{W}{g}v \cos \alpha$. That of the stream as it leaves the vane is $\frac{W}{g}v_1$, and its component upon the direction of the motion is $\frac{W}{g}v_1 \cos \theta$. The difference of these components is the total pressure in the given direction, or

$$P = \frac{W}{g}(v \cos \alpha - v_1 \cos \theta). \quad . \quad . \quad . \quad . \quad (85)$$

This is a general formula for the pressure in any given direction upon a vane moving in a straight line. If the surface be at rest v_1 equals v, and it agrees with the result deduced in Art. 130.

If it be required to find the numerical value of P, the given data are the velocities v and u, and the angles α and β. The term $v_1 \cos \theta$ is hence to be expressed in terms of these quantities. From the triangle at B between v_1 and u

$$v_1 \cos \theta = u - V \cos \beta.$$

Substituting this, the formula becomes

$$P = \frac{W}{g}(v \cos \alpha - u + V \cos \beta), \quad . \quad . \quad . \quad (85)'$$

which is often a more convenient form for discussion. The value of V is found from the triangle at A between u and v, thus:

$$V^2 = u^2 + v^2 - 2uv \cos \alpha ;$$

and hence the dynamic pressure P is fully determined in terms of the given data.

In order that the stream may enter tangentially upon the vane, and thus prevent foam, the curve of the vane at A should be tangent to the direction of V. This direction can be found by expressing the angle ϕ in terms of the given angle α. Thus from the relation between the sides and angles of the triangle included between u, v, and V there is found

$$\frac{\sin (\phi - \alpha)}{\sin \phi} = \frac{u}{v},$$

which reduces to the form

$$\cot \phi = \cot \alpha - \frac{u}{v \sin \alpha},$$

from which ϕ can be computed when u, v, and α are given. If the angle made by the vane with the direction of the motion be greater or less than ϕ some loss due to impact will result.

Prob. 159. What does v_1 represent in the parallelogram drawn at B in Fig. 93? Express its value in terms of β, u, and V.

ARTICLE 134. WORK DERIVED FROM MOVING VANES.

The work imparted to a moving vane by the energy of the impinging water is equal to the product of the dynamic pressure P, which is exerted in the direction of the motion and the space through which it moves. If u be the space described in one second, or the velocity of the vane, the work per second is

$$k = Pu.$$

This expression is now to be discussed in order to determine the value of u which makes k a maximum.

When the vane moves in a straight line in the same direction as the impinging jet and the water enters it tangentially, as shown in Fig. 87, the work imparted is found by inserting for P its value from (84), whence

$$k = (1 + \cos\beta)W\frac{(v-u)u}{g} = (1 + \cos\beta)wa\frac{(v-u)^2u}{g}.$$

The value of u which renders k a maximum is obtained by equating to zero the derivative of k with respect to u, or

$$\frac{\delta k}{\delta u} = (1 + \cos\beta)\frac{wa}{g}(v^2 - 4vu + 3u^2) = 0,$$

from which the value of u is

$$u = \tfrac{1}{3}v;$$

and inserting this, the maximum work is found to be

$$k = 8(1 + \cos\beta)\frac{wa}{27}\frac{v^2}{2g}.$$

The theoretic energy of the impinging jet is

$$K = W\frac{v^2}{2g} = wa\frac{v^3}{2g},$$

and accordingly the efficiency of the vane is (Art. 31)

$$e = \frac{k}{K} = \frac{8}{27}(1 + \cos\beta).$$

If $\beta = 180°$, the jet glides along the vane without producing work and $e = 0$; if $\beta = 90°$, the water departs from the vane normal to its original direction and $e = \frac{8}{27}$; if $\beta = 0$, the direction of the stream is reversed and $e = \frac{16}{27}$.

It appears from the above that the greatest efficiency which can be obtained by a vane moving in a straight line under the impulse of a jet of water is $\frac{16}{27}$; that is, the effective work is only about 59 per cent of the theoretic energy attainable.

This is due to two causes: first, the quantity of water which reaches and leaves the vane per second is less than that furnished by the nozzle or mouthpiece from which the water issues; and secondly, the water leaving the wane still has an absolute velocity of $\frac{1}{2}v$. A vane moving in a straight line is therefore a poor arrangement for utilizing energy, and it will also be seen upon reflection that it would be impossible to construct a motor in which a vane would move continually in the same direction away from a fixed nozzle. The above discussion therefore gives but a rude approximation to the results obtainable under practical conditions. It shows truly, however, that the efficiency of a jet which is deflected normally from its path is but one half of that obtainable when a complete reversal of direction is made.

Water wheels which act under the impulse of a jet consist of a series of vanes arranged around a circumference which by the motion are brought in succession before the jet. In this case the quantity of water which leaves the wheel per second is the same as that which enters it, so that W does not depend on the velocity of the vanes, as in the preceding case, but is a constant whose value is wq, where q is the quantity furnished per second. An approximate estimate of the efficiency of a series of such vanes can be made by considering a single vane and taking W as a constant. The water is supposed to impinge tangentially, and the vane to move in the same direction as the jet (Fig. 92). Then the work imparted per second is

$$k = (1 + \cos \beta) W \frac{(v - u)u}{g}.$$

This becomes zero when $u = 0$ or when $u = v$, and it is a maximum when $u = \frac{1}{2}v$, or when the vane moves with one-half the velocity of the jet. Inserting this value of u,

$$k = (1 + \cos \beta) W \frac{v^2}{4g};$$

and dividing this by the theoretic energy $W\frac{v^2}{2g}$, the efficiency is

$$e = \tfrac{1}{2}(1 + \cos \beta).$$

When $\beta = 180°$, the jet merely glides along the surface without performing work and $e = 0$; when $\beta = 90°$, the jet is deflected normally to the direction of the motion and $e = \frac{1}{2}$; when $\beta = 0°$, a complete reversal of direction is obtained and the efficiency attains its maximum value $e = 1$.

These conclusions apply approximately to the vanes of a water-wheel which are so shaped that the water enters upon them tangentially in the direction of the motion. If the vanes are plane radial surfaces, as in simply paddle-wheels, the water passes away normally to the circumference and the highest obtainable efficiency is about 50 per cent. If the vanes are curved backward the efficiency becomes greater, and, neglecting losses in impact and friction, it might be made nearly unity, and the entire energy of the stream be realized, if the water could both enter and leave the vanes in a direction tangent to the circumference. The investigation shows that this is due to the fact that the water leaves the vanes without velocity; for, as the advantageous velocity of the vane is $\frac{1}{2}v$, the water upon its surface has the relative velocity $v - \frac{1}{2}v = \frac{1}{2}v$; then, if $\beta = 0$, as it leaves the vane its absolute velocity is $\frac{1}{2}v - \frac{1}{2}v = 0$. If the velocity of the vanes is less or greater than half the velocity of the jet, the efficiency is lessened, although slight variations from the advantageous velocity do not practically influence the value of e.

Prob. 160. A nozzle 0.125 feet in diameter, whose coefficient of discharge is 0.95, delivers water under a head of 82 feet against a series of small vanes on a circumference whose diameter is 18.5 feet. Find the most advantageous velocity of revolution.

ARTICLE 135. REVOLVING VANES.

When vanes are attached to an axis around which they move, as is the case in water wheels, the dynamic pressure which is effective in causing the motion is that tangential to the circumferences of revolution; or at any given point this effective pressure is normal to a radius drawn from the point to the axis. In Fig. 94 are shown two cases of a rotating vane; in the first the water passes outward or away from the axis, and in the second it passes inward or toward the axis. The reasoning, however, is general and will apply to both cases. At A, where the jet enters upon the vane, let v be its absolute velocity, V its velocity relative to the vane, and u the velocity of the point A; draw u normal to the radius r and construct the parallelogram of velocities as shown, α being the angle between the directions of u and v, and ϕ that between u and V.

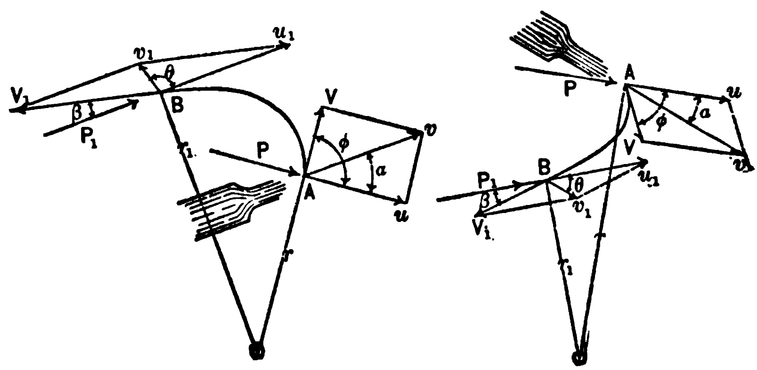

FIG. 94.

At B, where the water leaves the vane, let u_1 be the velocity of that point normal to the radius r_1, and V_1 the velocity of the water relative to the vane; then constructing the parallelogram, the resultant of u_1 and V_1 is v_1, the absolute velocity of the departing water. Let β be the angle between V_1 and the reverse direction of u_1, and θ be the angle between the directions of v_1 and u_1.

The total dynamic pressure exerted in the direction of the motion will depend upon the values of the impulse in the entering and departing streams. The absolute impulse of the water before entering is $\frac{W}{g}v$, and that of the water after leaving is $\frac{W}{g}v_1$. Let the components of these in the direction of the motion be designated by P and P_1; then,

$$P = \frac{W}{g}v \cos \alpha, \qquad P_1 = \frac{W}{g}v_1 \cos \theta.$$

These, however, cannot be subtracted to give the resultant dynamic pressure, as was done in the case of motion in a straight line, because their directions are not parallel, and the velocities of their points of application are not equal. The resultant dynamic pressure is not important in cases of this kind, but the above values will prove very useful in the next article in investigating the work that can be performed by the vane.

The given data for a revolving vane are the angles α and β, and the velocities v, u, and u_1. To find the auxiliary angle θ the triangle at B between u_1 and V_1 gives

$$v_1 \cos \theta = u_1 - V_1 \cos \beta.$$

When the motion is in a straight line the relative velocities V and V_1 are equal, if the friction is so slight that it can be neglected; for a revolving vane, however, they are unequal, and the relation between them will be deduced in the next article.

If n be the number of revolutions around the axis in one second, the velocities u and u_1 are

$$u = 2\pi rn, \qquad u_1 = 2\pi r_1 n,$$

and accordingly the relation obtains,

$$\frac{u}{u_1} = \frac{r}{r_1},$$

or the velocities of the points of entrance and exit are directly proportional to their distances from the axis. If r and r_1 are both infinity, u equals u_1 and the case is that of motion in a straight line as discussed in Art. 133.

Prob. 161. If a point 14 inches from the axis moves with a uniform velocity of 62 feet per second, how many revolutions does it make per minute?

ART. 136. WORK DERIVED FROM REVOLVING VANES.

The investigation in Art. 134 on the work and efficiency of a revolving vane supposes that all its points move with the same velocity, and that the water enters upon it in the same direction as that of its motion, or that $\alpha = 0$. This cannot in general be the case in water motors, as then the jet would be tangential to the circumference and no water could enter. To consider the subject further the reasoning of the last article will be continued, and, using the same notation, it will be plain that the work may be regarded as that due to the impulse of the entering stream in the direction of the motion around the axis minus that due to the impulse of the departing stream in the same direction, or

$$k = Pu - P_1u_1.$$

Here P and P_1 are the pressures due to the impulse at A and B (Fig. 94), and inserting their values as found,

$$k = \frac{W}{g}(uv \cos \alpha - u_1v_1 \cos \theta). \quad . \quad . \quad . \quad (86)$$

This is a general formula applicable to the work of all wheels of outward or inward flow, and it is seen that the useful work k consists of two parts, one due to the entering and the other to the departing stream.

Another general expression for the work of a series of vanes may be established as follows: Let v and v_1 be the absolute

velocities of the entering and departing water; then the theoretic energy is $W\frac{v^2}{2g}$, and there is carried away the energy $W\frac{v_1^2}{2g}$. The difference of these is the work imparted to the wheel, neglecting losses of energy in friction and impact, or

$$k = \frac{W}{2g}(v^2 - v_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

This is a formula of equal generality with the preceding, and like it is applicable to all cases of the conversion of energy into work by means of impulse or reaction. In both formulas, however, the plane of the vane is supposed to be horizontal, so that no fall occurs between the points of entrance and exit.

A useful relation between the relative velocities V and V_1 can be deduced by equating the values of k given by the preceding formulas; thus:

$$uv \cos \alpha - u_1v_1 \cos \theta = \tfrac{1}{2}(v^2 - v_1^2).$$

Now from the triangle at A between u and v

$$v^2 = V^2 - u^2 + 2uv \cos \alpha,$$

and from the triangle at B between u_1 and v_1

$$v_1^2 = V_1^2 - u_1^2 + 2u_1v_1 \cos \theta.$$

Inserting these values of v^2 and v_1^2 the relation reduces to

$$V_1^2 - u_1^2 = V^2 - u^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

This is the formula by which the relative velocity V_1 of the issuing water is to be computed when V is given. It shows when $u_1 = u$ that $V_1 = V$, as is the case in Fig. 93, where the motion is in a straight line. If, however, u_1 be greater than u, as in the outward-flow vane of the first diagram of Fig. 94, then V_1 is greater than V; if u_1 is less than u, as in an inward-flow vane, then V_1 is less than V.

The above principles will now be applied to the simple case

of a vane impinged upon tangentially by a jet which passes off in a radial direction. The two diagrams in Fig. 95 show the outward-flow and the inward-flow vane, and the reasoning will be general, and apply to both. As the velocity v of the jet has the same direction as the velocity of the vane, the relative velocity V at the point of entrance is $v - u$. The work imparted to the vane by the jet is, from (87),

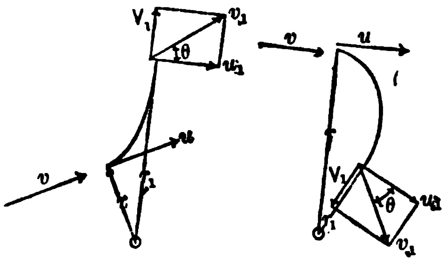

FIG. 95.

$$k = \frac{W}{2g}(v^2 - v_1^2).$$

From the parallelogram drawn at the point of exit,

$$v_1^2 = V_1^2 + u_1^2.$$

But from the relation established in (88),

$$V_1^2 = (v - u)^2 - u^2 + u_1^2,$$

whence v_1^2 is found to be

$$v_1^2 = v^2 - 2uv + 2u_1^2.$$

Hence the work imparted to the vane is

$$k = \frac{W}{g}(uv - u_1^2) = \frac{W}{g}\left(uv - u^2\frac{r_1^2}{r^2}\right).$$

This becomes zero when $u = 0$, or when $u = \frac{r^2}{r_1^2}v$; and it is a maximum when

$$u = \tfrac{1}{2}v\frac{r^2}{r_1^2}.$$

This advantageous velocity reduces the value of k to

$$k = \frac{W}{4g}\frac{r^2}{r_1^2}v^2.$$

If $r = 0$ the work k vanishes, as it should do; for this case is that of an outward-flow vane, where the water has only a radial motion. If $r = r_1$ the case is that of motion in a straight line, and k becomes one-half the theoretic energy of the jet, as in Art. 134. If $r_1 = 0$, the value of k becomes ∞; but this is absurd, since in no event can k be greater than $W\frac{v^2}{2g}$: the reason of this ridiculous conclusion lies partly in the fact that the assumption $r_1 = 0$ is an impossibility for an inward-flow vane, since the water must turn aside from the axis before reaching it, and partly in the circumstance that the advantageous value of u was deduced by supposing u_1 finite. Indeed, if $r^2 = 2r_1^2$, that value of u becomes v, and plainly u cannot exceed v if the water is to do work on the vane. The absurdity therefore may be said to be caused by the fact that the algebraic maximum for this case lies outside the limits of the problem.

Precisely the same conclusions may be drawn from the use of the formula (86) instead of (87); for since here $\alpha = 0$, and $v_1 \cos \theta = u_1$, it reduces to

$$k = \frac{W}{g}(uv - u_1^2),$$

which is the same as before found. It appears from this discussion that an outward-flow vane under the conditions of Fig. 95 cannot utilize more than one-half the energy of the jet, but that an inward-flow vane may utilize the entire energy. It is here again repeated, that the effect of friction and foam have been neglected in the investigation; these, of course, tend to make the velocity V_1 less than its theoretic value, and thus consume energy.

Prob. 162. In the first diagram of Fig. 94 let $\alpha = 0$, $\phi = 0$, $\beta = 0$, and $r = 0$. Prove that the advantageous velocity is infinite.

ARTICLE 137. REVOLVING TUBES.

The water which glides over a vane can never be under static pressure, but when two vanes are placed near together and connected so as to form a closed tube, there may exist in it static pressure if the tube is filled. This is the condition in turbine wheels, where a number of such tubes, or buckets, are placed around an axis and water is forced through them by the static pressure of a head. The work in this case is done by the dynamic pressure exactly as in vanes, but the existence of the static pressure renders the investigation more difficult.

The simplest instance of a revolving tube is that of an arm attached to a vessel rotating about a vertical axis, as in Fig. 96. It was shown in Art. 29 that the water surface in this case assumes the form of a paraboloid, and if no discharge occurs it is clear that the static pressures at any two points B and A are measured by the pressure-heads H_1 and H reckoned upwards to the parabolic curve, and, if the velocities of those points are u_1 and u, that

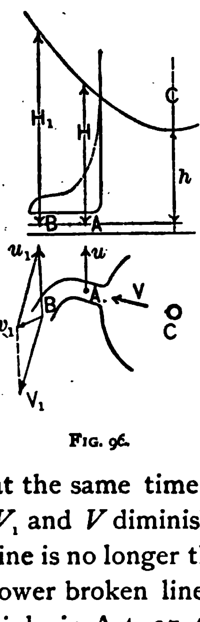

FIG. 96.

$$H_1 - \frac{u_1^2}{2g} = H - \frac{u^2}{2g} = h.$$

Now suppose an orifice to be opened in the end of the tube and the flow to occur while at the same time the revolution is continued. The velocities V_1 and V diminish the pressure-heads so that the piezometric line is no longer the parabola but some curve represented by the lower broken line in the figure. Then according to the principle in Art. 27, that pressure-head plus velocity-head remains constant if no loss of energy occurs, the above equation becomes

$$H_1 + \frac{V_1^2}{2g} - \frac{u_1^2}{2g} = H + \frac{V^2}{2g} - \frac{u^2}{2g}, \quad . \quad . \quad . \quad (89)$$

in which H_1 and H are the heads due to the actual static pressures. This is the theorem which gives the relation between pressure-head, velocity-head, and rotation-head at any point of a revolving tube or bucket. If the tube is only partly full, so that the flow occurs along one side, like that of a stream upon a vane, then there is no static pressure, $H_1 = 0$, $H = 0$, and the formula becomes the same as (88), which was otherwise deduced in the last article.

An apparatus like Fig. 96, but having a number of arms from which the flow issues, is called a reaction wheel, since the dynamic pressure which causes the revolution is wholly due to the reaction of the issuing water. To investigate it, the general formula (86) may be used. Making $u = 0$, then the work done upon the wheel by the water is

$$k = \frac{W}{g}(-\ u_1 v_1 \cos\theta) = \frac{W}{g}(u_1 V_1 \cos\beta - u_1^2).$$

But since there is no static pressure at the point B, the value of V_1 is, from (89), or also from Art. 29,

$$V_1 = \sqrt{2gh + u_1^2}.$$

The work of the wheel now is

$$k = \frac{W}{g}(u_1 \cos\beta \sqrt{2gh + u_1^2} - u_1^2).$$

This becomes nothing when $u_1 = 0$, or when $u_1^2 = 2gh \cot^2\beta$, and by the usual method it is found that it becomes a maximum when

$$u_1^2 = \frac{gh}{\sin\beta} - gh.$$

Inserting this advantageous velocity, the corresponding work is

$$k = Wh(1 - \sin\beta),$$

and therefore the efficiency is

$$e = 1 - \sin\beta.$$

When $\beta = 90°$, both u_1 and e become 0, for then the direction of the stream is normal to the circumference of revolution and no reaction can occur. When $\beta = 0$ the efficiency becomes unity, but the velocity u_1 becomes infinity. In the reaction wheel, therefore, high efficiency can only be secured by making the direction of the issuing water directly opposite to that of the revolution, and by having the speed very great. If $\beta = 19°.5$ or $\sin \beta = \frac{1}{3}$, the advantageous velocity u_1 becomes $\sqrt{2gh}$ and e becomes 0.67. The effect of friction of the water on the sides of the revolving tube is not here considered, but in Art. 143, where the reaction wheel is to be further discussed, this will be done.

Prob. 163. Compute the theoretic efficiency of the reaction wheel when $\theta = 180°$, $\beta = 0°$, and $u_1 = \sqrt{2gh}$.

CHAPTER XII.

HYDRAULIC MOTORS.

ARTICLE 138. CONDITIONS OF HIGH EFFICIENCY.

There are three ways in which water may act in imparting its energy to hydraulic motors, namely, by its weight, by the dynamic pressure of its impulse and reaction, and by its static pressure. To the first class belong those wheels where the water descends in buckets, to the second impulse wheels and turbines, and to the third those in which pistons are moved by static pressure. In some cases both weight and dynamic pressure act in the same motor, as in the breast wheel, and to a slight extent in the undershot. The following pages will be devoted to a discussion of some of the most important motors, in order to determine the conditions which render them most efficient.

The efficiency e of a motor ought, if possible, to be independent of the amount of water used, or if not, it should be the greatest when the water supply is low. This is very difficult to attain. It should be noted, however, that it is not the mere variation in the quantity of water which causes the efficiency to vary, but it is the losses of head which are consequent thereon. For instance, when water is low, gates must be lowered to diminish the area of orifices, and this produces sudden changes of section which diminish the effective head h. A complete theoretic expression for the efficiency will hence not include W, the weight of water supplied per second, but it should, if possible, include the losses of energy or head which result when W varies. The actual efficiency of a motor can

only be determined by tests with a friction brake; the theoretic efficiency, as deduced from formulas like those of the last chapter, will as a rule be higher than the actual, because it is impossible to formulate accurately all the sources of loss. Nevertheless, the deduction and discussion of formulas for theoretic efficiency is very important for the correct understanding and successful construction of hydraulic motors.

A general theoretic expression for the efficiency will now be deduced. The theoretic energy per second is

$$K = Wh = W\frac{v^2}{2g}.$$

The actual work per second equals the theoretic energy minus all the losses of energy. These losses may be divided into two classes: first, those caused by the transformation of energy into heat; and second, those due to the velocity v_1 with which the water reaches the level of the tail race. The first class includes losses in friction, losses in foam and eddies consequent upon sudden changes in cross-section, or from allowing the entering water to dash improperly against surfaces; let the loss of work due to this be Wh', in which h' is the head lost by these causes. The second loss is due merely to the fact that the departing water carries away the energy $W\frac{v_1^2}{2g}$. The work per second imparted to the wheel then is

$$k = W\left(h - h' - \frac{v_1^2}{2g}\right);$$

and dividing this by the theoretic energy, the efficiency is

$$e = 1 - \frac{h'}{h} - \left(\frac{v_1}{v}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

This formula, although very general, must be the basis of all discussions on the theory of water-wheels and motors. It shows that e can only become unity when $h' = 0$ and $v_1 = 0$,

whence the two following fundamental requirements must be fulfilled in order to secure high efficiency:

1. The water must enter and pass through the wheel without losing energy in friction and foam.
2. The water must reach the level of the tail race without absolute velocity.

These two requirements are expressed in popular language by the maxim, well known among engineers, "the water must enter the wheel without shock and leave without velocity." Here the word shock means that method of introducing the water which produces foam and eddies.

The friction of the wheel upon its bearings is included in the lost work when the power and efficiency are actually measured as described in Art. 124. But as this is not a hydraulic loss, it should not be included in the lost work k' when discussing the wheel merely as a user of water, as will be done in this chapter. The amount lost in shaft and journal friction in good constructions may be estimated at 2 or 3 per cent of the theoretic energy, so that in discussing the hydraulic losses the maximum value of e will not be unity, but about 0.98 or 0.97. This may perhaps be rendered slightly smaller by the friction of the wheel upon the air or water in which it moves, and which will here not be regarded.

Prob. 164. A wheel using 70 cubic feet per minute under a head of 12.4 feet has an efficiency of 0.63. What is its effective horse-power?

ARTICLE 139. OVERSHOT WHEELS.

In the overshot wheel the water acts largely by its weight. Fig. 97 shows an end view or vertical section, which so fully illustrates its action that no detailed explanation is necessary. The total fall from the surface of the water in the head race or flume to the surface in the tail race is called h. The weight of

water delivered per second is represented by W; then the theoretic energy of the fall per second is Wh. It is required to determine the conditions which will render the work of the wheel as near to Wh as possible.

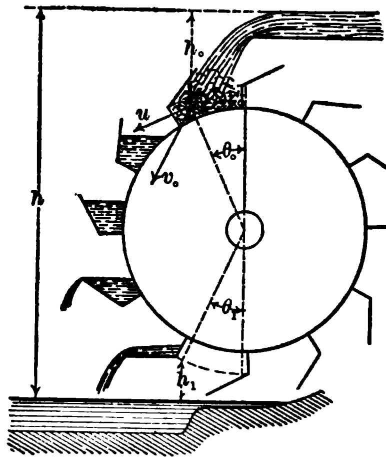

FIG. 97

The total fall may be divided into three parts—that in which the water is filling the buckets, that in which the water is descending in the filled buckets, and that which remains after the buckets are emptied. Let the first of these parts be called h_0, and the last h_1. In falling the distance h_0 the water acquires a velocity v_0 which is approximately equal to $\sqrt{2gh_0}$, and then striking the buckets this is reduced to u, the tangential velocity of the wheel, whereby a loss of energy in impact occurs. It then descends through the distance $h - h_0 - h_1$ acting by its weight alone, and finally dropping out of the buckets, reaches the level of the tail race with a velocity which causes a second loss of energy. Let h' be the head lost in entering the buckets, and let v_1 be the velocity of the water as it reaches the tail race. Then the efficiency of the wheel is given by the general formula (90), or

$$e = 1 - \frac{h'}{h} - \frac{v_1^2}{v^2};$$

and to apply it, the values of h' and v_1 are to be found. In this equation v is the velocity due to the head h, or $v = \sqrt{2gh}$.

The head lost when a stream of water with the velocity v_0 is enlarged in section so as to have the smaller velocity u, is, as proved in Art. 68,

$$h' = \frac{(v_0 - u)^2}{2g} = \frac{v_0^2 - 2v_0u + u^2}{2g}.$$

The velocity v_1 with which the water reaches the tail race depends upon the velocity u and the height h_1. Its energy as it leaves the buckets is $W\frac{u^2}{2g}$, and that required in the fall h_1 is Wh_1; the sum of these must be equal to the resultant energy, $W\frac{v_1^2}{2g}$, whence the value of v_1 is

$$v_1 = \sqrt{u^2 + 2gh_1}.$$

Inserting these values of h' and v_1 in the formula for e, and placing for v^2 its equivalent $2gh$, there is found

$$e = 1 - \frac{v_0^2 - 2v_0u + 2u^2 + 2gh_1}{2gh}.$$

The value of u which renders e a maximum is by the usual method found to be

$$u = \tfrac{1}{2}v_0;$$

or the velocity of the wheel should be one-half that of the entering water. Inserting this value, the efficiency corresponding to the advantageous velocity is

$$e = 1 - \frac{\frac{1}{2}v_0^2 + 2gh_1}{2gh};$$

and lastly, replacing v_0^2 by its value $2gh_0$, it becomes

$$e = 1 - \frac{1}{2}\frac{h_0}{h} - \frac{h_1}{h}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

which is the theoretic maximum efficiency of the overshot wheel.

This investigation shows that one-half of the entrance fall h_0 and the whole of the exit fall h_1 are lost, and it is hence plain that in order to make e as large as possible both h_0 and h_1 should be as small as possible. The fall h_0 is made small by making the radius of the wheel large; but it cannot be zero, for then no water would enter the wheel: it is generally taken so as to make the angle θ_0 about 10 or 15 degrees. The fall h_1 is made small by giving to the buckets a form which will retain the water as long as possible. As the water really leaves the wheel at several points along the lower circumference, the value of h_1 cannot usually be determined with exactness.

The practical advantageous velocity of the overshot wheel, as determined by the method of Art. 124, is found to be about $0.4v_0$, and its efficiency is found to be high, ranging from 70 to 90 per cent. In times of drought, when the water supply is low, and it is desirable to utilize all the power available, its efficiency is the highest, since then the buckets are but partly filled and h_1 becomes small. Herein lies the great advantage of the overshot wheel; its disadvantage is in its large size and the expense of construction and maintenance.

The number of buckets and their depth are governed by no laws except those of experience. Usually the number of buckets is about $5r$ or $6r$, if r is the radius of the wheel in feet, and their radial depth is from 10 to 15 inches. The breadth of the wheel parallel to its axis depends upon the quantity of water supplied, and should be so great that the buckets are not fully filled with water, in order that they may retain it as long as possible and thus make h_1 small. The wheel should be set with its outer circumference at the level of the tail water.

Prob. 165. Estimate the horse-power of an overshot wheel which uses 1080 cubic feet of water per minute under a head

of 26 feet, the diameter of the wheel being 23 feet, and the water entering at 15° from the top and leaving at 12° from the bottom.

ARTICLE 140. BREAST WHEELS.

The breast wheel is applicable to small falls, and the action of the water is partly by impulse and partly by weight. As represented in Fig. 98, water from a reservoir is admitted through an orifice upon the wheel under the head h_0 with the velocity v_0; the water being then confined between the vanes and the curved breast acts by its weight through a distance h_2,

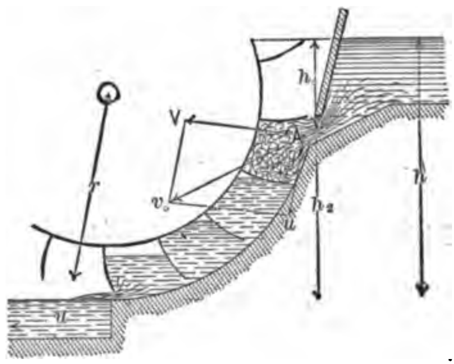

FIG. 98.

which is approximately equal to $h - h_0$, until finally it is released at the level of the tail race and departs with the velocity u, which is the same as that of the circumference of the wheel. The total energy of the water being Wh, the work of the wheel is eWh, if e be its efficiency.

The reasoning of the last article may be applied to the breast wheel, h_1 being made equal to zero, and the expression there deduced for e may be regarded as an approximate value of its theoretic efficiency. It appears, then, that e will be the greater the smaller the fall h_0; but owing to leakage between

the wheel and the curved breast, which cannot be theoretically estimated, and which is less for high velocities than for low ones, it is not desirable to make v_0 and h_0 small. The efficiency of the breast wheel is hence materially less than that of the overshot, and usually ranges from 50 to 80 per cent, the lower values being for small wheels.

Another method of determining the theoretic efficiency of the breast wheel is to discuss the action of the water in entering and leaving the vanes as a case of impulse. Let at the point of entrance Av_0 and Au be drawn parallel and equal to the velocities v_0 and u, the former being that of the entering water and the latter that of the vanes. Then the dynamic pressure exerted by the water in entering upon and leaving the vanes is, from Art. 133,

$$P = \frac{W}{g}(v_0 \cos \alpha - u),$$

and the work performed by it per second is

$$k_0 = \frac{W}{g}(v_0 \cos \alpha - u)u.$$

This is a maximum when

$$u = \tfrac{1}{2}v_0 \cos \alpha,$$

and the corresponding work of the impulse is

$$k_0 = \frac{W}{4g} v_0^2 \cos^2 \alpha.$$

Adding this to the work Wh_1 done by the weight of the water, the total work of the wheel when running at the advantageous velocity is

$$k = W\left(\frac{v_0^2 \cos^2 \alpha}{4g} + h_1\right);$$

or if v_0^2 be replaced by its value $c_1^2 \cdot 2gh_0$, where c_1 is the coefficient of velocity as determined by the rules of Chapters IV and VI,

$$k = W(\tfrac{1}{2}c_1^2 \cos^2 \alpha \,.\, h_0 + h_2),$$

whence the theoretic efficiency is

$$e = \tfrac{1}{2}c_1^2 \cos^2 \alpha \frac{h_0}{h} + \frac{h_2}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

If in this expression h_2 be replaced by $h - h_0$, and if $c_1 = 1$ and $\alpha = 0°$, this reduces to the same value as found for the overshot wheel. The angle α, however, cannot be zero, for then the direction of the entering water would be tangential to the wheel, and it could not impinge upon the vanes; its value, however, should be small, say from 10° to 25°. The coefficient c_1 is to be rendered large by making the orifice of discharge with well-rounded inner corners so as to avoid contraction and the losses incident thereto. The above formulas cannot be relied upon in practice to give close values of k and e, on account of losses by foam and leakage along the curved breast, which of course cannot be algebraically expressed.

Prob. 166. A breast wheel is 10.5 feet in diameter, and has $c_1 = 0.93$, $h_0 = 4.2$ feet, and $\alpha = 12$ degrees. Compute the most advantageous number of revolutions per minute.

ARTICLE 141. UNDERSHOT WHEELS.

The common undershot wheel has plane radial vanes, and the water passes beneath it in a direction nearly horizontal. It may then be regarded as a breast wheel where the action is entirely by impulse, so that in the preceding equations h_2 becomes 0, h_0 becomes h, and α will be 0°. The theoretic efficiency then is

$$e = \tfrac{1}{2}c_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)'$$

In the best constructions c_1 is nearly unity, so that it may be concluded that the maximum efficiency of the undershot wheel is about 0.5. Experiment shows that its actual efficiency varies from 0.20 to 0.40, and that the advantageous velocity is about $0.4v_0$ instead of $0.5v_0$. The lowest efficiencies are obtained from wheels placed in an unlimited flowing current, as upon a scow anchored in a stream; and the highest from those where the stream beneath the wheel is confined by walls so as to prevent the water from spreading laterally.

The Poncelet wheel, so called from its distinguished inventor, has curved vanes, which are so arranged that the water leaves them tangentially, with its absolute velocity less than that of the velocity of the wheel. If in Fig. 98 the fall h_2 be very small, and the vanes be curved more than represented, it will exhibit the main features of the Poncelet wheel. The water entering with the absolute velocity v_0 takes the velocity u of the vane and the velocity V relative to the vane. Passing then under the wheel, its dynamic pressure performs work; and on leaving the vane its relative velocity V is probably nearly the same as that at entrance. Then if V be drawn tangent to the vane at the point of exit, and u tangent to the circumference, their resultant will be v_1, the absolute velocity of exit, which will be much less than u. Consequently the energy carried away by the departing water is less than in the usual forms of breast and undershot wheels, and it is found by experiment that the efficiency may be as high as 60 per cent.

When water is delivered through a nozzle against the vanes of a wheel in a direction nearly tangent to the lowest point of the circumference, the action is entirely similar to that of the undershot wheel. Such machines are called vertical impulse wheels, or (improperly) impact wheels; they are extensively used in California under the name of "hurdy-gurdy wheels," which are arranged so as to be easily transported from place

to place, as may be necessary in mining operations, the nozzle being attached to a hose or pipe which brings the water from a canal. Fig. 99 shows an outline sketch of such a wheel. The simplest vanes are radial planes as at A, but these are found to give a low efficiency. Curved vanes, as at B, are also used with much better results, as they cause the water to turn backward opposite to the direction of the motion, and thus to leave with a low absolute velocity (Art. 134). In the plan of the wheel it is seen that the vanes may be arranged so as to turn the water sidewise while deflecting it backward. The experiments made by BROWNE* show that with plane radial vanes the highest efficiency was 40.2 per cent, while with curved vanes or cups 82.5 per cent was obtained. The velocity of the vanes which gave the highest efficiency was in each case found to be a little less than one-half the velocity of the jet. The hurdy-gurdy is a fast-running wheel, as the stream issuing from the nozzle has commonly a high velocity.

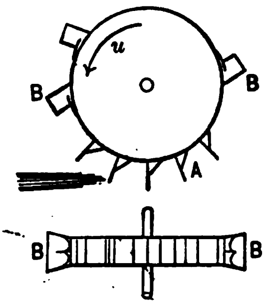

FIG. 99.

Prob. 167. The diameter of a hurdy-gurdy wheel is 12.58 feet between centres of vanes, and the impinging jet has a velocity of 58.5 feet per second and a diameter of 0.182 feet. The efficiency of the wheel is 44.5 per cent when making 62 revolutions per minute. What horse-power does it furnish?

Ans. 4.09 H. P.

ARTICLE 142. HORIZONTAL IMPULSE WHEELS.

Figure 100 represents portions of the circumference of two horizontal wheels, driven by the dynamic pressure of a stream of water issuing from a nozzle. In one the water enters the

* BOWIE's Practical Treatise on Hydraulic Mining, p. 193.

wheel upon the inner and leaves it upon the outer circumference, and in the other the reverse is the case. The first form is hence called an outward-flow, and the second an inward-flow, wheel. The water issuing from the nozzle with the velocity v impinges upon the vanes, and in passing through the wheel alters both its direction and its absolute velocity, thus transforming its energy into useful work. The energy of the entering water is $W\frac{v^2}{2g}$, and that of the departing water is $W\frac{v_1^2}{2g}$, if v_1 be

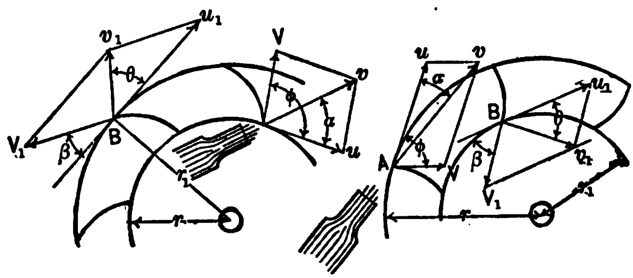

FIG. 100.

its absolute velocity. The work imparted to the wheel then is

$$k = W\left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right),$$

and dividing this by the theoretic energy, the efficiency is

$$e = 1 - \frac{v_1^2}{v^2}.$$

This is the same as the general formula (90) if $k' = 0$, that is, if losses in foam and friction are disregarded, and if the wheel is set at the level of the tail race. It is now required to find an expression for v_1^2, whose discussion will determine the conditions for securing the greatest efficiency. The reasoning will be general and applicable to both outward- and inward-flow wheels.

At the point A where the water enters the wheel let the

parallelogram of velocities be drawn, the absolute velocity of entrance being resolved into its two components, the velocity u of the wheel at that point, and the velocity V relative to the vane; let α be the angle between u and v, and ϕ be the angle between u and V. At the point B where the water leaves the wheel let V_1 be its velocity relative to the vane, and u_1 the velocity of the wheel at that point; then their resultant is v_1, the absolute velocity of exit. Let β be the angle between V_1 and the reverse direction of u_1, and θ the angle between u_1 and v_1. The directions of the velocities u and u_1 are of course tangential to the circumferences at the points A and B. Let r and r_1 be the radii of these circumferences; then the velocities of revolution are directly as the radii, or $ur_1 = u_1r$.

Now to determine the value of v_1^2 the triangle at B between u_1 and v_1 gives

$$v_1^2 = u_1^2 + V_1^2 - 2u_1V_1 \cos \beta.$$

The value of V_1 is found by the formula (88) of Art. 136, namely,

$$V_1^2 = V^2 - u^2 + u_1^2.$$

The value of V^2 from the triangle at A between u and v is

$$V^2 = u^2 + v^2 - 2uv \cos \alpha.$$

Hence the first equation becomes

$$v_1^2 = v^2 - 2uv \cos \alpha + 2u_1^2 - 2u_1 \cos \beta \sqrt{v^2 - 2uv \cos \alpha + u_1^2}.$$

Substituting for u_1 its value in terms of u, and placing the resultant value of v_1^2 in the expression for the efficiency, there is found

$$e = 2\frac{u}{v} \cos \alpha - 2\frac{u^2}{v^2} \cdot \frac{r_1^2}{r^2}$$
$$+ 2\,\frac{u}{v}\,\frac{r_1}{r} \cos \beta \sqrt{1 - 2\frac{u}{v} \cos \alpha + \frac{u^2}{v^2}\,\frac{r_1^2}{r^2}} \quad . \quad . \quad . \quad (93)$$

This is the general formula for the efficiency of a horizontal-

impulse wheel, and it will now be discussed in order to determine what values of α, β, u, and $\frac{r_1}{r}$ are most advantageous.

The value of u which renders e a maximum cannot be determined in a form which is available for use, since the derivative equated to zero leads to a biquadratic equation. But when $\cos\beta = 0$, or $\beta = 90°$, the advantageous velocity is

$$u = \frac{1}{2}\frac{r^2}{r_1^2} v \cos\alpha,$$

as also shown in Art. 136, and for an inward-flow wheel e can be made nearly unity. It is also seen from Fig. 100 that the only way in which v_1 can be made 0 is when $u_1 = V_1$ and $\beta = 0°$. It is hence a good rule that β should be a small angle, but this is more important in an outward- than in an inward-flow wheel. The formula shows also that e increases when $\cos\beta$ and $\cos\alpha$ increase, and hence in general both β and α ought to be small angles.

If the angle β be not large, the condition $u_1 = V_1$ will give a near approach to the conditions of best efficiency. When this occurs, $u = V$, and the advantageous velocity is

$$u = \frac{v}{2\cos\alpha}.$$

This reduces the formula for the efficiency to

$$e = 1 - \frac{1}{2}\frac{r_1^2}{r^2}\frac{1 - \cos\beta}{\cos^2\alpha}. \quad . \quad . \quad . \quad . \quad (93)'$$

It is clearly shown by this formula that in an inward-flow wheel, where r_1 is small compared with r, e may have a high value even if α and β are not very small angles. For instance, let $r \doteq 5r_1$, $\alpha = 30°$, and $\beta = 45°$; then

$$e = 1 - 0.5 \times 0.04 \times \tfrac{4}{3}(1 - 0.707) = 0.992.$$

For an outward-flow wheel, however, where r_1 is larger than r, it is absolutely necessary that β should be small.

The actual efficiency of horizontal-impulse wheels is rarely greater than 75 per cent. There is, of course, a material loss of energy in foam when the water enters the wheel, and in friction as it passes through it. To reduce the foam as much as possible, the direction of the vanes at the entrance circumference should correspond with the relative velocity V. If the advantageous velocity u is known, the angle ϕ which determines this direction is given by the rule established in Art. 133, namely,

$$\cot \phi = \cot \alpha - \frac{u}{v \sin \alpha}.$$

If the wheel be run at the velocity $u = \dfrac{v}{2 \cos \alpha}$, this becomes

$$\cot \phi = \cot 2\alpha,$$

and hence ϕ is double the angle α. It is usual to make ϕ somewhat greater than 2α, however, and it is often made 90° when α is less than 30°. For this practice there seems to be no good reason.

Prob. 168. If the wheel be run at such a velocity that v_1 is radial, or $\theta = 90°$, deduce expressions for u and e, and show that the efficiency is less than in the above case where $u_1 = V_1$.

ARTICLE 143. REACTION WHEELS.

The reaction wheel, sometimes called Barker's mill, consists of a number of hollow arms connected with a hollow vertical shaft, as shown in Fig. 101. The water issues from the ends of the arms in a direction opposite to that of their motion, and by the dynamic pressure due to its reaction the energy of the water is transformed into useful work. Let the head of

water CC in the shaft be h; then the pressure-head BB which causes the flow from the arms is greater than h, on account of centrifugal force. Let V_1 be the velocity of discharge relative to the wheel; then, as shown in Art. 29,

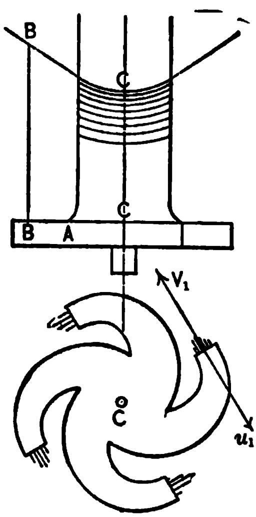

FIG. 101.

$$V_1 = \sqrt{2gh + u_1^2}.$$

The absolute velocity v_1 of the issuing water now is

$$v_1 = V_1 - u_1 = \sqrt{2gh + u_1^2} - u_1.$$

It is seen at once that the efficiency can never reach unity unless $v_1 = 0_1$, which requires that $V_1 = u_1$. This, however, can only occur when $u_1 = \infty$, since the above formula shows that V_1 must be greater than u_1 for any finite values of h and u_1. To deduce an expression for the efficiency the general formula (90) may be used, placing for v_1^2 its value $(V_1 - u_1)^2$, and for v^2 its value $2gh$ or $V_1^2 - u_1^2$; then

$$e = 1 - \frac{(V_1 - u_1)^2}{V_1^2 - u_1^2} = \frac{2u_1}{V_1 + u_1} \quad . \quad . \quad . \quad . \quad . \quad (94)$$

This shows, as before, that e equals unity when $V_1 = u_1 = \infty$. If $V_1 = 2u_1$ the value of e is 0.667; if $V_1 = 3u_1$, the value of e is 0.50.

The above formula may be deduced in another way, as follows: The absolute velocity of the issuing water being $V_1 - u_1$, the dynamic pressure of its reaction is

$$P = \frac{W}{g}v_1 = \frac{W}{g}(V_1 - u_1),$$

and the work done by this pressure is

$$k = Pu_1 = \frac{W}{g}(V_1 - u_1)u_1.$$

The theoretic energy of the water is

$$K = Wh = \frac{W}{2g}(V_1^2 - u_1^2),$$

and accordingly the efficiency is

$$e = \frac{k}{K} = \frac{2u_1}{V_1 + u_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

This investigation apparently shows, although e approaches unity as u_1 approaches V_1, that the effective work k decreases. This is due to the fact that W, as above written, is regarded as constant, which is not the case, as its value is wa_1V_1. Hence, really, as u_1 increases so does W, and when $u_1 = V_1 = \infty$, the value of k is Wh; but as W is then also ∞, the work would be unlimited. Practically, of course, none of these conditions can be approached, so that the full theoretic efficiency of the reaction wheel can never be realized.

To consider the effect of friction in the arms, let c_1 be the coefficient of velocity (Chapter VI), so that

$$V_1 = c_1\sqrt{2gh + u_1^2}.$$

Then the effective work of the wheel is

$$k = \frac{W}{g}(c_1u_1\sqrt{2gh + u_1^2} - u_1^2),$$

and the efficiency is

$$e = \frac{c_1u_1\sqrt{2gh + u_1^2} - u_1^2}{gh}.$$

The value of u_1 which renders this a maximum is

$$u_1^2 = \frac{gh}{\sqrt{1 - c_1^2}} - gh,$$

and this reduces the value of the efficiency to

$$e = 1 - \sqrt{1 - c_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)'$$

If $c_1 = 1$, there is no loss in friction, and $u_1 = \infty$ and $e = 1$, as before deduced. If $c_1 = 0.94$, the advantageous velocity u_1 is very nearly $\sqrt{2gh}$, and e is 0.66; hence the influence of fric-

tion in diminishing the efficiency is very great. In order to make c_1 large, the end of the arm where the water enters must be well rounded to prevent contraction, and the interior surface must be smooth. If the inner end has sharp square edges, as in a standard tube (Art. 61), c_1 is 0.82, and e becomes about 0.43.

The reaction wheel is not now used as a hydraulic motor on account of its low efficiency. Even when run at high speeds the efficiency is low on account of the greater friction and resistance of the air. By experiments on a wheel one meter in diameter under a head of 1.3 feet WEISBACH* found a maximum efficiency of 67 per cent when the velocity of revolution u_1 was $\sqrt{2gh}$. When u_1 was $2\sqrt{2gh}$ the efficiency was nothing, or all the energy was consumed in frictional resistances. By reasoning like that of Art. 137 it may be shown when the issuing water makes an angle β with the line of motion, as in Fig. 96, that

$$e = 1 - \sqrt{1 - c_1^2 \cos^2 \beta}, \quad . \quad . \quad . \quad . \quad . \quad (94)''$$

and that the corresponding advantageous speed is given by the formula above if c_1 be replaced by $c_1 \cos \beta$.

If the water issuing from the arms impinge upon planes firmly attached to the arms, as at M in Fig. 102, no motion occurs, for the water is then deflected in a radial direction, and it exerts no dynamic pressure which can cause revolution. If, however, it impinge in hemispherical cups attached to the arms which deflect the water backward, the motion occurs in a contrary direction to that of the common reaction wheel, for the velocity u_1 must be opposite to the absolute velocity $V_1 - u_1$. Lastly, it may be men

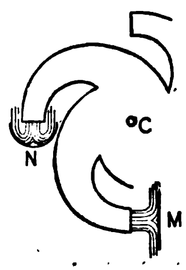

FIG. 102.

* Hydraulics and Hydraulic Motors, DUBOIS's translation, p. 385.

tioned that the common reaction wheel when placed under water sometimes runs in a direction opposite to that of its usual motion. This is probably due to the circumstance that there is no static pressure in front of the orifice where the stream issues; so that, if the depth of immersion be sufficient, the static pressure behind it may be greater than the dynamic pressure of the reaction, and accordingly the resultant pressure is in the direction of the flow. There are, in fact, few machines which illustrate so many of the laws of hydraulics as this marvellous reaction wheel.

Prob. 169. Compute the effective power of a reaction wheel when $\beta = 0°$, $h = 16$ feet, $u_1 = \sqrt{2gh}$, $c_1 = 0.95$, $r_1 = 1.75$ feet, and $a_1 = 4.25$ square inches, the latter being the sum of the areas of the exit orifices. Ans. 1.59 H.P.

ARTICLE 144. FLOW THROUGH TURBINE WHEELS.

A turbine wheel acts under the dynamic pressure of flowing water which at the same time may be under a certain degree of static pressure. If in the reaction wheel of Fig. 96 the arms be separated from the penstock at A, and be so arranged that BA revolves around the axis while AC is stationary, the resulting apparatus may be called a turbine. The static pressure of the head CC can still be transmitted through the arms, so that, as in the reaction wheel, the discharge will be influenced by the speed of rotation. The general arrangement of entrance and exit angles, however, is like the impulse wheel, the dynamic pressure being due to reaction only in a slight degree. Turbine wheels are now used more extensively than other hydraulic motors, on account of their cheapness, compactness, and high efficiency. The different kinds may be classified as outward flow, inward flow, downward flow, and those in which the flow is partly inward and partly downward.

In Fig. 103 are shown horizontal and vertical sections of the outward- and inward-flow types, without the inclosing case. The moving wheel marked W consists of a series of vanes set in a frame which is attached by arms to the central axis; the spaces between these vanes will be called buckets. The water is brought to the buckets by a series of guides set in a fixed frame G, and it is seen that the water is introduced around the entire circumference of the wheel. Between the guides and the wheel is an annular space in which slides an annular verti-

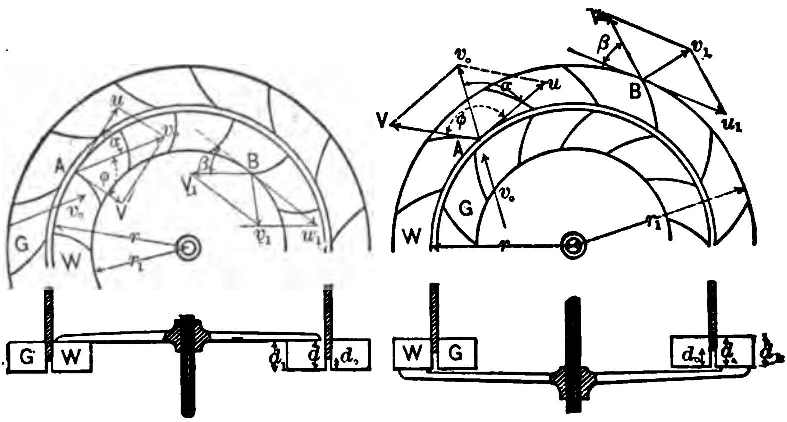

FIG. 103.

cal gate; this stops the admission of water when entirely depressed, and serves to regulate the quantity furnished. The spaces between the guides, as also the buckets, are usually entirely filled with water when the wheel is in motion, and such will be taken to be the case in the following discussions.

A formula for the discharge q through a turbine wheel when the gate is fully raised is now to be established. Let h be the head between the water levels in the penstock and tail race, H_1 the pressure-head on the exit orifices or the depth of the latter below the tail water level, and H the pressure-head

at the gate opening as indicated by a piezometer supposed to be there inserted as seen in Fig. 104. Let u_1 and u be the velocities of the wheel at the exit and entrance circumference, whose radii are r_1 and r (Fig. 103). Let V_1 and V be the relative velocities of exit and entrance, and v_0 be the absolute velocity of the water as it leaves the guides and enters the wheel; v_0 may be less or greater than $\sqrt{2gh}$, depending upon the value of the pressure-head H. Let a_1, a, and a_0 be the areas of the orifices normal to the directions of V_1, V, and v_0. Now, neglecting all losses of friction between the guides, the theorem of Art. 27, that pressure-head plus velocity-head equals the total head, gives

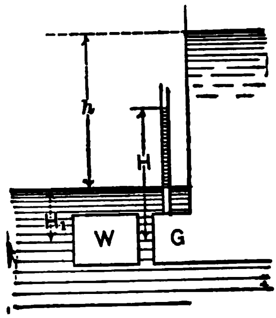

FIG. 104.

$$H + \frac{v_0^2}{2g} = h + H_1.$$

Also, neglecting the friction and foam in the buckets, the theorem of Art. 137 gives

$$H_1 + \frac{V_1^2}{2g} - \frac{u_1^2}{2g} = H + \frac{V^2}{2g} - \frac{u^2}{2g}.$$

Adding these equations, the pressure-heads H_1 and H disappear and there results the formula

$$V_1^2 - V^2 + v_0^2 = 2gh + u_1^2 - u^2.$$

Now, since the buckets are fully filled, the same quantity of water, q, passes in each second through each of the areas a_1, a, and a_0, and

$$V_1 = \frac{q}{a_1}, \qquad V = \frac{q}{a}, \qquad v_0 = \frac{q}{a_0}.$$

Introducing these values of the velocity, solving for q, and

multiplying by a coefficient c to account for losses in leakage and friction, the discharge per second is

$$q = c\sqrt{\frac{2gh + u_1^2 - u^2}{\frac{1}{a_1^2} - \frac{1}{a^2} + \frac{1}{a_0^2}}}. \quad . \quad . \quad . \quad . \quad (95)$$

This is the formula for the flow through a turbine in motion, when the gate is fully raised. In an outward-flow turbine u_1 is greater than u, and consequently the discharge increases with the speed; in an inward-flow turbine u_1 is less than u, and consequently the discharge decreases as the speed increases.

The value of the coefficient c will probably vary with the head, and also with the size of the areas a_1, a, and a_0. For the outward-flow Boyden turbine, the tests of which are given in Art. 126, it lies between 0.94 and 0.95, as the following results show, where the first four columns contain the number of the experiment, the observed head, number of revolutions per minute, and discharge in cubic feet per second. The fifth column gives the theoretic discharge computed from the above formula, taking the coefficient as unity, and the last column is

No.	h.	N.	q.	Q.	c.
21	17.16	63.5	117.01	123.1	0.950
20	17.27	70.0	118.37	125.2	0.945
19	17.33	75.0	119.53	126.8	0.943
18	17.34	80.0	121.15	128.4	0.944
17	17.21	86.0	122.41	130.0	0.942
16	17.21	93.2	124.74	132.5	0.941
15	17.19	100.0	127.73	134.9	0.947

derived by dividing the observed discharge q by the theoretic discharge Q. The discrepancy of 5 or 6 per cent is smaller than might be expected, since the formula does not consider frictional resistances.

A satisfactory formula for the discharge through a turbine

when the gate is partly depressed is difficult to deduce, because the loss of head which then results can only be expressed by the help of experimental coefficients similar to those given in Art. 75 for the sliding gate in a water pipe, and the values of these for turbines are not known. It is, however, certain that for each particular gate opening the discharge is given by

$$q = m\sqrt{2gh + u_1^2 - u^2}; \quad . \quad . \quad . \quad . \quad . \quad (95)'$$

in which m depends upon the areas of the orifices and the height to which the gate is raised. For instance, in the tests of the Boyden turbine of Art. 126, the value of m is 2.815 when the proportional gate opening is 0.609, and the computed discharges will differ in no case more than one per cent from those observed; when the proportional gate opening is 0.200, the value of m is 1.357. And each turbine will have its own values of m, depending upon the area of its orifices.

A downward-flow turbine is one in which each particle of water remains at the same distance from the axis in its path through the guides and buckets. In Fig. 105 is seen a semi-vertical section of the wheel, and also a development of a por-

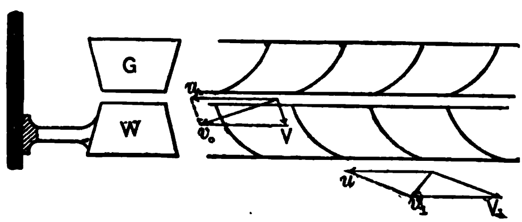

FIG. 105.

tion of a cylindrical section showing the inner arrangement. The formula for the discharge can be adapted to this by mak. $u_1 = u$. In this turbine there is no action of centrifugal force, so that the relative exit velocity V_1 is equal to V, or at the most equal to $\sqrt{V^2 + 2gh_0}$, where h_0 is the vertical depth of the wheel.

The three typical classes of turbines above described are often called by the names of those who first invented or perfected them; thus the outward-flow is called the Fourneyron, the inward-flow the Francis, and the downward-flow the Jonval, turbine. There are also many turbines in the market in which the flow is a combination of inward and downward motion, the water entering horizontally and inward, and leaving vertically; the bucket partitions in these are warped surfaces, and the angle β is not the same at all points of exit. These wheels of combined direction seem on the whole to be those which give the promise of ultimately attaining the highest efficiency. The usual efficiency of turbines at full gate is from 70 to 85 per cent, although 90 per cent has in some cases been derived. When the gate is partly closed the efficiency in general decreases, and when the gate opening is small it becomes very low, as the example in Art. 126 shows. This is due to the loss of head consequent upon the sudden change of cross-section; and therein lies the disadvantage of the turbine, for when the water supply is low, it is important that the wheel should utilize all the power available. That this difficulty will ultimately be overcome there can be little doubt.*

Prob. 170. Deduce the value of the coefficient m for the experiments Nos. 1–7 in Art. 126, and then compare the computed with the observed discharges.

ARTICLE 145. THEORY OF TURBINES.

A volume could easily be written upon the theory of turbines alone, and hence the brief space here available must be limited to the most important topic among the many which might be discussed, namely, the conditions of maximum efficiency. This may be divided into two problems: first, to

* Concerning a wheel whose efficiency is highest at about three-fourths gate, see Transactions American Society Mechanical Engineers, vol. viii. p. 359.

determine the advantageous velocity, and the corresponding efficiency, for a turbine whose dimensions are known; and second, to ascertain how a turbine should be built so that the greatest efficiency can be obtained. The investigations will be limited to the case where the gate is fully open. In both problems there are four sources of loss of energy which should be carefully kept in mind: first, loss in the absolute velocity of the departing water; second, loss in friction; third, loss in foam; fourth, loss in leakage between the guides and the wheel.

The first problem may be discussed by imposing the condition that the absolute velocity v_1 of the departing water should be as small as possible. This will occur, if β be not a large angle, when $u_1 = V_1$, as then their resultant v_1 becomes very small. Now V_1 may be expressed in terms of the discharge q and the area a_1; thus the condition for the minimum is

$$u_1 = \frac{q}{a_1}.$$

Substituting the value of q from (95), and solving for u_1, there is found

$$u_1^2 = \frac{c^2 \cdot 2gh}{1 + c^2 \dfrac{r^2}{r_1^2} + \dfrac{a_1^2}{a_0^2} - \dfrac{a_1^2}{a^2} - c^2}, \quad . \quad . \quad . \quad . \quad (96)$$

which gives the advantageous velocity of the circumference where the water leaves the wheel. When $u_1 = V_1$, the velocity-square of the departing water is

$$v_1^2 = 2u_1^2(1 - \cos\beta) = 4u_1^2 \sin^2 \tfrac{1}{2}\beta;$$

and accordingly the efficiency of the turbine is

$$e = 1 - \frac{4c^2 \sin^2 \frac{1}{2}\beta}{1 + c^2 \dfrac{r^2}{r_1^2} + \dfrac{a_1^2}{a_0^2} - \dfrac{a_1^2}{a^2} - c^2} - \frac{h'}{h}, \quad . \quad . \quad . \quad (96)'$$

in which the last term represents the losses due to friction, foam,

and leakage. These losses can scarcely be formulated, and as their sum is often greater than the loss due to the energy of the departing water, which is represented by the second term, it is plain that the determination of the theoretic efficiency in any particular case cannot often be successfully made.

The second problem, to design a turbine so that the efficiency may be a maximum at full gate, can be also approximately solved by making $u_1 = V_1$, and by adding the condition that there should be no leakage. Resuming the first and second equations of the last article, it is seen from Fig. 104 that there will be no leakage if $H = H_1$. The two conditions hence reduce the second equation to $u = V$, and the first to $v_0^2 = 2gh$. But when $u = V$, the angle ϕ should be 2α, in order that the water may enter the buckets tangentially. Also $q = a_1V_1 = a_1u_1$, and $q = aV = au$; hence a_1u_1 equals au, or a_1r_1 equals ar. Therefore, to prevent leakage and foam, the turbine should be so built that

$$\phi = 2\alpha, \quad \text{and} \quad a_1r_1 = ar. \quad . \quad . \quad . \quad . \quad (97)$$

These, however, give maximum efficiency only when the speed of the wheel corresponds to the condition $u = V$; inserting for V its value in terms of v_0 and α, this furnishes

$$u = \frac{v_0}{2 \cos \alpha} = \frac{1}{2} \sec \alpha \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad (97)'$$

which is the advantageous velocity of the circumference where the water enters. For the other circumference

$$u_1 = u\frac{r_1}{r} = \frac{1}{2}\frac{r_1}{r} \sec \alpha \sqrt{2gh},$$

and then the maximum efficiency is

$$e = 1 - \frac{r_1^2}{r^2} \sec^2 \alpha \sin^2 \frac{1}{2}\beta - \frac{h'}{h}, \quad . \quad . \quad . \quad (97)''$$

in which the last term refers to frictional losses only, since leakage and foam are avoided by the construction. The friction of the water on the guides and vanes may be estimated from general knowledge regarding flow in tubes to consume about 5 per cent, and for the axle friction there may be put 2 per cent; thus the least value of h' is probably about $0.07h$.

It is seen from the preceding discussions that the efficiency increases as α and β decrease, but that β is more important than α; for if β in (97)″ be 0, the angle α may have any value less than a right angle, and the term containing it will vanish. It is likewise seen that e increases as the ratio $r_1 \div r$ decreases, that is, an inward-flow turbine is preferable to one of outward flow; and if this ratio can be very small, the angle β need not necessarily be small. The areas a_1, a, and a_0, above used, are those normal to the directions of the velocities V_1, V, and u_0; they can be expressed in terms of the radii r_1 and r, and the vertical depths d_1 and d, as follows:

$$a_1 = 2\pi r_1 d_1 \sin \beta, \quad a = 2\pi r d \sin \phi, \quad a_0 = 2\pi r d \sin \alpha.$$

Using these values, the second condition in (97) may be otherwise expressed by

$$r_1^2 d_1 \sin \beta = r^2 d \sin 2\alpha.$$

From this equation, after assuming the angles and radii, the depths d_1 and d can be arranged. The above values of the areas can also, if desired, be inserted in the formulas (96) and (97). It is customary in testing a turbine to measure these areas directly, but in making a design the above expressions will be found useful.

Referring again to the outward-flow turbine of Art. 126, it is seen that neither of the conditions of (97) are fulfilled; for ϕ is 90° and α is 24°, while $a_1 r_1 = 15.3$ and $ar = 32.3$. There existed, therefore, a loss by leakage under the gate, and a loss due

to foam and enlargement of section at entrance into the wheel. By formula (96)′ it is found that the loss due to the energy of the departing water was about 11 per cent. If the frictional losses were about 7 per cent, the losses due to leakage and foam amounted to about 5 per cent, since the observed efficiency was 77 per cent. The value of u_1 in experiment 20 was 24.3 feet per second, and that of V_1 was 25.7, so that the condition $u_1 = V_1$ was approximately fulfilled. For experiment 19, the value of u_1 is 26.0 and that of V_1 is 25.9, so that for some speed intermediate between those of the two experiments the condition would have been exactly satisfied. It is hence indicated that the assumption of deducing the advantageous velocity from the minimum of v_1^2 is entirely correct.

Prob. 171. Show that in experiment 20, above alluded to, the velocity v_0 was 24.9 feet per second; that the pressure-head $H - H_1$, which caused leakage, was 7.64 feet.

ARTICLE 146. OTHER KINDS OF MOTORS.

Machines in which the static pressure of water acts upon pistons have been used to some extent, particularly for small motors, although in Europe under the name of water-pressure engines many large ones have been built. Fig. 106 shows in a diagrammatic way the method of their action, in which are seen two reservoirs A and B, the head between them being h. When the valves marked M are open and those marked N are closed water passes from the upper to the lower reservoir, and the piston moves in the direction of the arrow. As the piston reaches the end of its stroke, the valves M are closed while N are opened; the piston then reverses its motion, and water passes from A to B through the other set of pipes. In the practical construction of the apparatus it is necessary that the

pipes should be large, so that the velocity of flow may be small in order to render the frictional resistances as slight as possible, and to avoid the shocks which would result when the valves close. The reservoir B may be omitted, so that the water from the cylinder discharges through the lower valves directly into the air, and arrangements may be devised so that but one pipe from the upper reservoir is needed. In any event the principle of action is like that of the steam-engine, except that there can be no cut-off until the piston reaches the end of its stroke. If frictional resistances could be entirely avoided, the theoretic efficiency of this motor might be made very high, for the work done in one stroke is wAh if A be the area of the piston, and in one second is $nwAh$ if n be the number of strokes per second. But nwA is the weight of water delivered per second, and thus the work done is Wh, which is the theoretic energy of the fall.

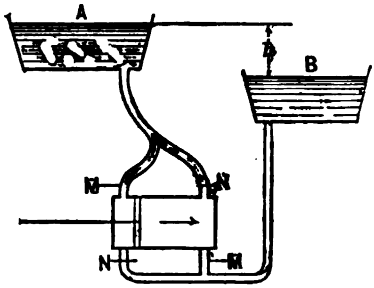

FIG. 106.

The screw wheel consists of one or two turns of a helicoidal surface around a vertical shaft, the screw being inclosed in a cylindrical case which prevents the water from escaping. The downward pressure of the water can then be resolved into two components: one parallel to the surface, which causes a relative velocity V; and one horizontal, which corresponds to the velocity of the wheel. This apparatus has never come into practical use as a motor, and probably for the reason that, like the reaction wheel, an infinite velocity of revolution is theoretically necessary in order to secure maximum efficiency.

The old-style tub wheel is a horizontal impulse wheel which consists of a series of buckets into which a stream from a spout

impinges in an inclined direction. They are not efficient, and hence are now little used. The same remark may be made concerning many other devices which have from time to time been used as hydraulic motors. The wheels described in the preceding pages are almost the only ones now in use, and certainly the only ones whose use is extensive. The turbine now leads all the others on account of its small size, cheapness, applicability to both large and small falls, and high efficiency. In this improvements are constantly being made, and undoubtedly it is the wheel of the future.

Prob. 172. How can a turbine be set 30 feet above the level of the tail race, and still secure the power due to the total head?

CHAPTER XIII.

NAVAL HYDROMECHANICS.

ARTICLE 147. GENERAL PRINCIPLES.

In this chapter is to be discussed in a brief and elementary manner the subject of the resistance of water to the motion of vessels, and the general hydrodynamic principles relating to their propulsion. The water may be at rest and the vessel in motion—or both may be in motion, as in the case of a boat going up or down a river. In either event the velocity of the vessel relative to the water need only be considered, and this will be called v. The simplest method of propulsion is by the oar or paddle; then come the paddle wheel, and the jet and screw propellers. The action of the wind upon sails will not be here discussed, as lying outside of the scope of the work.

The unit of measure used on the ocean is generally the nautical mile or knot, which is about 6080 feet, so that knots per hour may be transformed into feet per second by multiplying by 1.69, and feet per second may be transformed into knots per hour by multiplying by 0.592. On rivers the statute mile is used, and the corresponding multipliers will be 1.47 and 0.682. On the ocean the weight of a cubic foot of water is to be taken as about 64 pounds (it is often used as 64.32 pounds, so that the numerical value is the same as $2g$), and in rivers at 62.5 pounds.

The speed of a ship at sea is roughly measured by observations with the log, which is a triangular piece of wood attached to a cord which is divided by tags into lengths of about $50\frac{2}{3}$

feet. The log being thrown, the number of tags run out in half a minute is the same as the number of knots per hour at which the ship is moving, since $50\frac{2}{3}$ feet is the same part of a knot that a half minute is of an hour. The patent log, which is a small self-recording current meter, drawn in the water behind the ship, is however now generally used. In experimental work more accurate methods of measuring the velocity are necessary, and for this purpose the boat may run between buoys whose distance apart has been found by triangulation from measured bases on shore.

When a boat or ship is to be propelled through water, the resistances to be overcome increase with its velocity, and consequently, as in railroad trains, a practical limit of speed is soon attained. These resistances consist of three kinds—the dynamic pressure caused by the relative velocity of the boat and the water, the frictional resistance of the surface of the boat, and the wave resistance. The first of these can be entirely overcome, as indicated in Art. 132, by giving to the boat a "fair" form, that is, such a form that the dynamic pressure of the impulse near the bow is balanced by that of the reaction of the water as it closes in around the stern. It will be supposed in the following pages that the boat has this form, and hence this first resistance need not be further considered. The second and third sources of resistance will be discussed later.

The total force of resistance which exists when a vessel is propelled with the velocity v can be ascertained by drawing it in tow at the same velocity, and placing on the tow line a dynamometer to register the tension. An experiment by FROUDE on the Greyhound, a steamer of 1157 tons, gave for the total resistance the following figures: *

At 4 knots per hour,	0.6 tons;
At 6 knots per hour,	1.4 tons;

* THEARLE'S Theoretical Naval Architecture, London, 1876, p. 347.

At 8 knots per hour, 2.5 tons;
At 10 knots per hour, 4.7 tons;
At 12 knots per hour, 9.0 tons.

This shows that at low speeds the resistance varies about as the square of the velocity, and at higher speeds in a faster ratio. For speeds of 15 to 18 knots per hour—the usual velocity of ocean steamers—there is but little known regarding the resistance, but as an approximation it is usually taken as varying with the square of the velocity.

Prob. 173. What horse-power was expended in the above test of the Greyhound when the speed was 12 knots per hour?

ARTICLE 148. FRICTIONAL RESISTANCES.

When a stream or jet moves over a surface its velocity is retarded by the frictional resistances, or if the velocity be maintained uniform a constant force is overcome. In pipes, conduits, and channels of uniform section the velocity is uniform, and consequently each square foot of the surface or bed exerts a constant resisting force, the intensity of which will now be approximately computed. This resistance will be the same as the force required to move the same surface in still water, and hence the results will be directly applicable to the propulsion of ships.

Let F be the force of frictional resistance per square foot of surface of the bed of a channel, p its wetted perimeter, l its length, h its fall in that length, a the area of its cross-section, and v the mean velocity of flow. The force of friction over the entire surface then is Fpl, and the work per second lost in friction is $Fplv$. The work done by the water per second is Wh or $wavh$. Equating these two expressions for the work,

$$F = w\frac{ah}{pl} = wrs.$$

Now, inserting for rs its value from formula (70) of Art. 94, there results

$$F = \frac{w}{c^2}v^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

in which w is the weight of a cubic foot of water, and c is the coefficient in the mean velocity formulas whose value is to be taken from the tables in Chapter VIII. Inasmuch as the velocities along the bed of a channel are somewhat less than the mean velocity v, the values of F thus determined will probably be slightly greater than the actual resistance.

For smooth iron pipes the following are values of the frictional resistance in pounds per square foot of surface at different velocities, as computed from the above formula:

	$v =$	2.	4.	6.	10.	15.
For 1 foot diameter,	$F =$	0.023,	0.080,	0.17,	0.43,	0.92;
For 4 feet diameter,	$F =$	0.015,	0.053,	0.11,	0.28,	0.59.

These figures indicate that the resistance is subject to much variation in pipes of different diameters; it is not easy to conclude from them, or from formula (98), what the force of resistance is for plane surfaces over which water is moving.

Experiments made by moving flat plates in still water so that the direction of motion coincides with the plane of the surface have furnished conclusions regarding the laws of fluid friction similar to those deduced from the flow of water in pipes. It is found that the total resistance is approximately proportional to the area of the surface, and approximately proportional to the square of the velocity. Accordingly, the force of resistance per square foot may be written

$$F = fv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)'$$

in which f is a number depending upon the nature of the sur-

face. The following are average values of f for large surfaces, as given by UNWIN:*

Varnished surface,	$f = 0.00250$;
Painted and planed plank,	$f = 0.00339$;
Surface of iron ships,	$f = 0.00351$;
Fine sand surface,	$f = 0.00405$;
New well-painted iron plate,	$f = 0.00473$.

Undoubtedly the value of f is subject to variations with the velocity, but the experiments on record are so few that the law and extent of its variation cannot be formulated. It should, however, be remarked that the formulas and constants here given do not apply to low velocities, for the reasons given in Art. 92. At the same time they are only approximately applicable to high velocities A low velocity of a body moving in an unlimited stream may be regarded as 1 foot per second or less, a high velocity as 25 or 30 feet per second.

It may be noted that the above-mentioned experiments indicate that the value of F is greater for small surfaces than for large ones. For instance, a varnished board 50 feet long gave $f = 0.00250$, while one 20 feet long gave $f = 0.00278$, and one 8 feet long gave $f = 0.00325$, the motion being in all cases in the direction of the length. The resistance is the same whatever be the depth of immersion, for the friction is uninfluenced by the intensity of the static pressure. This is proved by the circumstance that the flow of water in a pipe is found to depend only upon the head on the outlet end, and not upon the pressure-heads along its length.

Prob. 174. What is the frictional resistance of a boat when moving at the rate of 9 knots per hour, the area of its immersed surface being 320 square feet, and $f = 0.0035$?

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* Encyclopædia Britannica, 9th Edition, vol. xii. p. 483.

ARTICLE 149. WORK REQUIRED IN PROPULSION.

When a boat or ship moves through still water with a velocity v, it must overcome the pressure due to impulse of the water and the resistance due to the friction of its surface on the water and air. If the surface be properly curved there is no resultant pressure due to impulse, as shown in Art. 132. The resistance caused by friction of the immersed surface on the water can be estimated, as explained above. If A be the area of this surface in square feet, the work per second required to overcome this resistance is

$$k = AFv = fAv^3. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (99)$$

The work, and hence the horse-power, required to move a boat accordingly varies approximately as the cube of its velocity. By the help of the values of f given in the last article an approximate estimate of the work can be made for particular cases. The resistance of the air, which in practice must be considered, will be here neglected.

To illustrate this law let it be required to find how many tons of coal will be used by a steamer in making a trip of 3000 miles in 6 days, when it is known that 800 tons are used in making the trip in 10 days. As the power used is proportional to the amount of coal, and as the distances travelled per day in the two cases are 500 miles and 300 miles, the law gives

$$\frac{T}{800} = \frac{5^3}{3^3},$$

whence $T = 3700$ tons. By the increased speed the expense for fuel is increased 463 per cent, while the time is reduced 40 per cent. If the value of wages, maintenance, interest, etc., saved on account of the reduction in time, will balance the extra expense for fuel, the increased speed is profitable. That such a compensation occurs in many instances is apparent from

the constant efforts to reduce the time of trips of passenger steamers.

When a boat moves with the velocity v in a current which has a velocity u in the same direction the velocity of the boat relative to the water is $v - u$, and the resistance is proportional to $(v - u)^2$ and the work to $(v - u)^3$. If the boat moves in the opposite direction to the current the relative velocity is $v + u$, and of course v must be greater than u or no progress would be made. In all cases of the application of the formulas of this article and the last, v is to be taken as the velocity of the boat relative to the water.

Another source of resistance to the motion of boats and ships is the production of waves. This is due in part to a different level of the water surface along the sides of the ship due to the variation in static pressure caused by the velocity, and in part to other causes. It is plain that waves, eddies, and foam cause energy to be dissipated in heat, and that thus a portion of the work furnished by the engines of the boat is lost. This source of loss is supposed to consume from 10 to 40 per cent of the total work, and it is known to increase with the velocity. On account of the uncertainty regarding this resistance, as well as those due to the friction of the water and air, practical computations on the power required to move boats at given velocities can only be expected to furnish approximate results.

Prob. 175. Compute the horse-power required for a velocity of 18 knots per hour, taking $A = 7473$ square feet and $f = 0.004$.

ARTICLE 150. THE JET PROPELLER.

The method of jet propulsion consists in allowing water to enter the boat and acquire its velocity, and then to eject it backwards at the stern by means of a pump. The reaction thus

produced propels the boat forward. To investigate the efficiency of this method, let W be the weight of water ejected per second, V its velocity relative to the boat, and v the velocity of the boat itself. The absolute velocity of the issuing water is then $V - v$, and it is plain without further discussion that the maximum efficiency will be obtained when this is 0, or when $V = v$, as then there will be no energy remaining in the water which is propelled backward. It is, however, to be shown that this condition can never be realized.

The work which is lost in the absolute velocity of the water is

$$k' = \frac{W}{2g}(V - v)^2.$$

The work which is exerted on the boat by the reaction is

$$k = \frac{W}{g}(V - v)v.$$

The sum of these is the total theoretic work, or

$$K = \frac{W}{2g}(V^2 - v^2).$$

Therefore the efficiency of jet propulsion is expressed by

$$e = \frac{k}{K} = \frac{2v}{V + v}. \quad . \quad . \quad . \quad . \quad . \quad (100)$$

This becomes equal to unity when $v = V$ as before indicated, but then it is seen that the work k becomes 0 unless W is infinite. The value of W is waV, if a be the area of the orifices through which the water is ejected; and hence in order to make e unity and at the same time perform work it is necessary that either V or a should be infinity. The jet propeller is therefore like a reaction wheel (Art. 143), and it is seen upon comparison that the formula for efficiency is the same in the two cases.

By equating the above value of the useful work to that established in the last article there is found

$$fgAv^3 = waV(V - v);$$

and if this be solved for V, and the resulting value be substituted in (100), it reduces to

$$e = \frac{4}{3 + \sqrt{1 + \frac{4fgA}{wa}}},$$

which again shows that e approaches unity as the ratio of a to A increases. The area of the orifices of discharge must hence be very large in order to realize both high power and high efficiency. For this reason attempts to propel vessels by this method have not proved practically successful. In nature the same result is seen, for no marine animal except the cuttle-fish uses this principle of propulsion. Even the cuttle-fish cannot depend upon his jet to escape from his enemies, but for this relies upon his supply of ink, with which he darkens the water about him.

Prob. 176. Compute the approximate area of the orifices for a jet propeller to run at 10 knots per hour when exerting 1200 horse-power, the efficiency being 0.67.

ARTICLE 151. PADDLE WHEELS.

The method of propulsion by rowing and paddling is familiar to all. The power is furnished by muscular energy within the boat, the water is the fulcrum upon which the blade of the oar acts, and the force of reaction thus produced is transmitted to the boat and urges it forward. If water were an unyielding substance, the theoretic efficiency of the oar should be unity, or, as in any lever, the work done by the force at the rowlock should equal the work performed by the motive force exerted. But as the water is yielding, some of it is driven backward by the blade of the oar, and thus energy is lost.

The paddle or side wheel so extensively used in river navigation is similar in principle to the oar. The former is furnished

by a motor within the boat, the blades or vanes of the wheel tend to drive the water backward, and the reaction thus produced urges the boat forward. On first thought it might be supposed that the efficiency of the method would be governed by laws similar to those of the undershot wheel, and such would be the case if the vessel were stationary and the wheel were used as an apparatus for moving the water. In fact, however, the theoretic efficiency of the paddle wheel is much higher than that of the undershot motor.

The work exerted by the steam-engine upon the paddle wheels may be represented by PV, in which P is the pressure produced by the vanes upon the water, and V is their velocity of revolution; and the work actually imparted to the boat may be represented by Pv, in which v is its velocity. Accordingly the efficiency of the paddle, neglecting losses due to foam and waves, is

$$e = \frac{v}{V} = \frac{v}{v + v_1},$$

in which v_1 is the difference $V - v$, or the so-called "slip." If the slip be 0, the velocities V and v are equal, and the theoretic efficiency is unity. The value of V is determined from the radius r of the wheel and its number of revolutions per minute; thus $V = 2\pi rN$.

On account of the lack of experimental data it is difficult to give information regarding the practical efficiency of paddle wheels considered from a hydromechanic point of view. Owing to the water which is lifted by the blades, and to the foam and waves produced, much energy is lost. They are, however, very advantageous on account of the readiness with which the boat can be stopped and reversed. When the wheels are driven by separate engines, as is sometimes done on river boats, perfect control is secured, as they can be revolved in opposite directions when desired. Paddle wheels with feathering blades

are more efficient than those with fixed radial ones, but practically they are found to be cumbersome, and liable to get out of order.* In ocean navigation the screw has now almost entirely replaced the paddle wheel on account of its higher efficiency.

Prob. 177. Ascertain the size of the paddle wheels of the steamship Great Eastern.

ARTICLE 152. THE SCREW PROPELLER.

The screw propeller consists of several helicoidal blades attached at the stern of a vessel to the end of a horizontal shaft which is made to revolve by steam power. The dynamic pressure of the reaction developed between the water and the helicoidal surface drives the vessel forward, the theoretic work of the screw being the product of this pressure by the distance traversed. The pitch of the screw is the distance, parallel to the shaft, between any point on a helix, and the corresponding point on the same helix after one turn around the axis, and the pitch may be constant at all distances from the axis, or it may be variable. If the water were unyielding, the vessel would advance a distance equal to the pitch at each revolution of the shaft; actually, the advance is less than the pitch, the difference being called the slip. The effect thus is that the pressure P existing between the helical surfaces and the water moves the vessel with the velocity v, while the theoretic velocity which should occur is V, being the pitch of the screw multiplied by the number of revolutions per second. The work expended is hence PV or $P(v + v_1)$, if v_1 be the slip per second, and the work utilized is Pv. Accordingly the efficiency of screw propulsion is, approximately,

$$e = \frac{v}{v + v_1},$$

* For description of these, see KNIGHT's Mechanical Dictionary.

which is the same expression as before found for the paddle wheel. Here, as in the last article, all the pressure exerted by the blades upon the water is supposed to act backward in a direction parallel to the shaft of the screw, and the above conclusion is approximate because this is actually not the case, and also because the action of friction has not been considered.

The pressure P which is exerted by the helicoidal blades upon the water is the same as the thrust or stress in the shaft, and the value of this may be approximately ascertained by regarding it as due to the reaction of a stream of water of cross-section a and velocity v_1, or

$$P = \frac{wa}{g}(v + v_1)v_1 .$$

Another expression for this may be found from the expended work k; thus:

$$P = \frac{k}{v_1}.$$

Numerical values computed from these two expressions do not, however, agree well, the latter giving in general a much less value than the former.

In Art. 149 the work to be performed in propelling a vessel of fair form whose submerged surface is A was found to be

$$k = fAv^3.$$

If the value of v is taken from this and inserted in the expression for efficiency, there obtains

$$e = \frac{1}{1 + v_1\left(\frac{Af}{k}\right)^{\frac{1}{3}}},$$

which shows that e increases as v_1, f, and A decrease, and as k increases. Or for given values of f and A the efficiency decreases with the speed.

It has been observed in a few instances that the slip v_1 is negative, or that V, as computed from the number of revolutions and pitch of the screw, is less than v. This is probably due to the circumstance that the water around the stern is following the vessel with a velocity v', so that the real slip is $V - v + v'$ instead of $V - v$. The existence of negative slip is usually regarded as evidence of poor design.

In some cases twin screws are used, as with these the vessel can be more readily controlled. Fig. 107 shows the twin screws of the City of New York, an ocean steamer of 580 feet

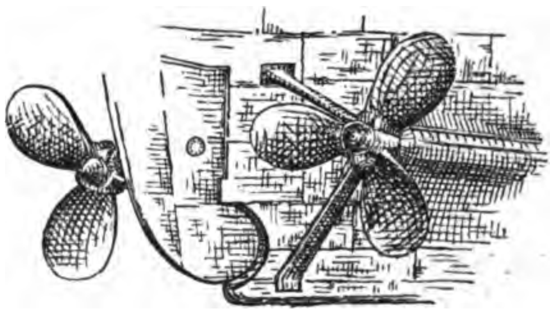

FIG 107.

length, 63.5 feet breadth, and 42 feet depth, with a gross tonnage of 10 500 and an estimated horse-power of about 16 000. These are made to revolve in opposite directions. The usual practice, however, is to have but one screw. The practical advantage of the screw over the paddle wheel has been found to be very great, and this is probably due to the circumstance that less energy is wasted in lifting the water and in forming waves.

Prob. 178. Ascertain the size and the pitch of the screws on the steamer City of New York. Compute the theoretic efficiency if the number of revolutions per minute is 150 when the velocity of the steamer is 20 knots per hour.

ARTICLE 153. THE ACTION OF THE RUDDER.

The action of the rudder in steering a vessel involves a principle that deserves discussion. In Fig. 108 is shown a plan of a boat with the rudder turned to the starboard side, at an angle θ with the line of the keel. The velocity of the vessel being v, the action of the water upon the rudder is the same as if the vessel were at rest and the water in motion with the velocity v.

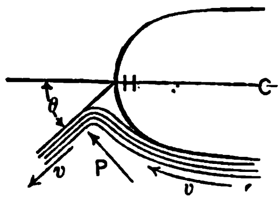

FIG. 108.

Let W be the weight of water which produces dynamic pressure against the rudder, due to the impulse $\frac{W}{g}v$ (Art. 128). The component of this pressure normal to the rudder is

$$P = \frac{W}{g} v \sin \theta,$$

and its effect in turning the vessel about the centre of gravity C is measured by its moment with reference to that point. Let b be the breadth of the rudder, and d the distance CH between the centre of gravity and the hinge of the rudder; then the lever arm of the force P is

$$l = \tfrac{1}{2}b + d \cos \theta,$$

and accordingly the turning moment is

$$M = \frac{W}{2g} v (b \sin \theta + d \sin 2\theta).$$

To determine that value of θ which produces the greatest effect in turning the boat the derivative of M with respect to θ must vanish, which gives

$$\cos \theta = -\frac{b}{8d} + \sqrt{\frac{1}{2} + \frac{b^2}{64d^2}},$$

and from this the value of θ is found to be approximately 45°, since d is always much larger than b.

The following are values of θ for several values of the ratio $b \div d$:

$d \div b =$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{100}$	0
$\cos\theta =$	0.6825	0.6916	0.6947	0.7069	0.7071
$\theta =$	46° 58′	46° 15′	46° 00′	45° 01′	45°

In practice it is usual to arrange the mechanism of the rudder so that it can only be turned to an angle of about 42° with the keel, for it is found that the power required to turn it the additional 3° or 4° is not sufficiently compensated by the slightly greater moment that would be produced. The reasoning also shows that intensity of the turning moment increases with v, so that the rudder acts most promptly when the boat is moving rapidly. For the same reason a rudder on a steamer propelled by a screw does not need to be so broad as on one driven by paddle wheels, for the effect of the screw is to increase the velocity of the impinging water, and hence also its dynamic pressure against the rudder.

Prob. 179. Explain how it is that a ship can sail against the wind.

ARTICLE 154. TIDES AND WAVES.

The complete discussion of the subject of waves might, like so many other branches of Hydraulics, be expanded so as to embrace an entire treatise, and hence there can be here given only the briefest outline of a few of the most important principles. There are two classes or kinds of waves, the first including the tidal waves and those produced by earthquakes or other sudden disturbances, and the second those due to the wind. The daily tidal wave generated by the attraction of the moon and sun originates in the South Pacific Ocean, whence

it travels in all directions with a velocity dependent upon the depth of water and the configuration of the continents, and which in some regions is as high as 1000 miles per hour. Striking against the coasts, the tidal waves cause currents in inlets and harbors, and if the circumstances were such that their motion could become uniform and permanent, these might be governed by the same laws which apply to the flow of water in channels. Such, however, is rarely the case; and accordingly the subject of tidal currents is one of much complexity, and not capable of general formulation.

The velocity of a wave produced by a sudden disturbance in a channel of uniform width is found by experiments, and also by theoretic considerations, to be $\sqrt{gD}$, where D designates the depth of water. When such a wave advances into shallow water its height is observed to increase, and when D becomes as small as one-half the height of the wave it breaks into foam.

Rolling waves produced by the wind travel with a velocity which is small compared with those of the first class, although in water where the disturbance can extend to the bottom it is generally supposed that their speed is also represented by

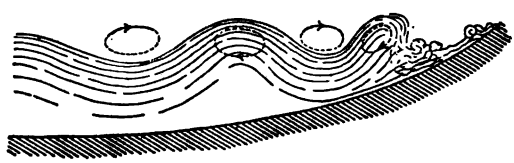

FIG. 109.

$\sqrt{gD}$. Upon the ocean the maximum length of such waves is estimated at 550 feet, and their velocity at about 53 feet per second. For this class of waves it is found by observation that each particle of water upon the surface moves in an elliptic or circular orbit, whose time of revolution is the same as the time

of one wave length. Thus the particles on the crest of a wave are moving forward in the direction of the motion of the wave, while those in the trough are moving backward. When such waves advance into shallow water their length and speed decrease, but the time of revolution of the particles in their orbits remains unaltered, and as a consequence the slopes become steeper and the height greater, until finally the front slope becomes vertical, and the wave breaks with roar and foam. Below the surface the particles revolve also in elliptic orbits, which grow smaller in size toward the bottom. The curve formed by the vertical section of the surface of a wave at right angles to its length is of a cycloidal nature.

The force exerted by ocean waves when breaking against sea walls is very great, as already mentioned in Art. 132, and often proves destructive. If walls can be built so that the waves are reflected without breaking, as is sometimes possible in deep water, their action is rendered less injurious. Upon the ocean waves move in the same direction as the wind, but along shore it is observed that they move normally toward it, whatever may be the direction in which the wind is blowing.

Prob. 180. In a channel 6.5 feet wide, and of a depth decreasing uniformly 1.5 feet per 1000, BAZIN generated a wave by suddenly admitting water at the upper end. At points where the depths were 2.16, 1.85, 1.46, and 0.80 feet, the velocities were observed to be 8.70, 8.67, 7.80, and 6.69 feet per second. Do these velocities agree with the law above stated?

APPENDIX.

Answers to Problems.

Below will be found the answers to most of the problems whose solution is not stated in the text, the number of the problem being enclosed in parenthesis. A few answers have been purposely omitted in order that the student may be thrown entirely upon his own resources. However satisfactory it may be to know in advance the result of the solution of an exercise, let the student bear in mind that after commencement day answers to problems will not be given him.

Chapter I. (1), 996, or, in round numbers, 1000 kilos per square centimeter. (3), 393 pounds, 47.1 gallons, 178 kilos. (5), 73.8 pounds, 5.02 atmospheres. (9), 14.73 pounds. (10), 7.85 gallons, 65.6 pounds.

Chapter II. (12), 100 feet, 100 meters. (16), 5280 pounds. (17), 2880 feet. (18), 8.04 feet, 2000 pounds. (19), $y = \frac{1}{3}d$. (22), 8.7 feet and 8.4 feet. (23), 8.7 pounds.

Chapter III. (27), $v = 8.75$ and 2.19 feet per second. (29), 0.0534 cubic feet per second. (33), 15.3 and 19.6 cubic feet per second. (34), 90.1 feet per second. (35), 94.6 feet per second. (37), $\frac{8}{7}\sqrt{2}$ times that for the hemisphere. (39), 318. (42), 1.81 horse-powers.